ORDINARY DIFFERENTIAL EQUATIONS

1. (a) We solve this equation by separation of variables. For $\alpha = 1$,

$$\int \frac{\mathrm{d}y}{y} = \ln y = \int \mathrm{d}x = x + C$$

for some $C \in \mathbb{R}$, so $y = e^C e^x$. The initial condition $y(0) = 0$ cannot be satisfied.

(b) When $\alpha \neq 1$, we proceed as above to arrive at

$$\frac{1}{-\alpha+1} y^{-\alpha+1} = x + C.$$

The general solution is therefore given by $y = [(-\alpha+1)(x+C)]^{\frac{1}{-\alpha+1}}$. Subject to the initial condition $y(0) = 0$, we deduce that $C = 0$, thus

$$y = ((1-\alpha)x)^{\frac{1}{1-\alpha}}.$$

(c) For the above solution to exist, it is necessary that $(1-\alpha)x \geq 0$; in other words, we require that $\alpha < 1$.

When $\alpha = -1$, we have $y = \pm\sqrt{2x}$. More generally, for any α of the form $\alpha = -(2k-1)$ where k is a positive integer, two solutions exist.

2. For each equation we use the substitution $y = e^{\lambda x}$ and solve the corresponding characteristic polynomial for λ. Using the solutions for λ, we construct the general solutions of the equations as presented in the lectures.

(a) The characteristic polynomial is given by $2\lambda^2 + 7\lambda - 4 = 0$, giving $\lambda_1 = \frac{1}{2}$, and $\lambda_2 = -4$. The general solution to this homogenous equation is therefore

$$y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{-4x}.$$

(b) The characteristic polynomial is given by $\lambda^2 + 2\lambda + 1 = 0$, so $\lambda_1 = \lambda_2 = -1$. The general solution is therefore

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

(c) The characteristic polynomial is given by

$$\lambda^3 - \lambda^2 - 9\lambda + 9 = (\lambda - 1)(\lambda^2 - 9) = 0,$$

leading to the solutions $\lambda_1 = -3$, $\lambda_2 = 3$, $\lambda_3 = 1$. Hence

$$y(x) = c_1 e^{-3x} + c_2 e^{3x} + c_3 e^x.$$

3. We first solve the corresponding homogeneous equation $y'' + y' - 6y = 0$ with characteristic polynomial
$$\lambda^2 + \lambda - 6\lambda = (\lambda - 2)(\lambda + 3) = 0.$$
The solutions are $\lambda_1 = 2$ and $\lambda_2 = -3$, so
$$y_h(x) = c_1 e^{2x} + c_2 e^{-3x}.$$
For a particular solution of the inhomogeneous equation, we use the ansatz $y_p(x) = Ce^x$. Then
$$y'' + y' - 6y = Ce^x + Ce^x - 6Ce^x = -4Ce^x.$$
This is equal to $-2\alpha e^x$ for $C = \frac{\alpha}{2}$, hence $y_p(x) = \frac{\alpha}{2}e^x$. In conclusion,
$$y(x) = y_h(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-3x} + \tfrac{\alpha}{2}e^x,$$
which is bounded if and only if $c_1 = \alpha = 0$. The general bounded solution is thus given by
$$y(x) = c_2 e^{-3x}.$$

4. First, consider the homogeneous equation $y'' + 2y' + 2y = 0$. The characteristic polynomial is $\lambda^2 + 2\lambda + 2 = 0$ with solutions $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$; in other words complex conjugate roots. The general solution of the homogeneous ODE is therefore in both cases
$$y_h(t) = e^{-t}(c_1 \cos t + c_2 \sin t).$$
We construct a particular solution for both cases simultaneously by using the fact that $\cos t = \mathrm{Re}(e^{it})$ and $\sin t = \mathrm{Im}(e^{it})$. Thus, it suffices to determine a particular (complex) solution of
$$y'' + 2y' + 2y = e^{it},$$
then extract the real and imaginary parts. We use the ansatz $y_p(t) = Ce^{it}$, such that
$$-C + 2iC + 2C = 1$$
when substituting in the above equation and cancelling the common factor e^{it}. In particular $C = \frac{1-2i}{5}$, so
$$y_p(t) = \tfrac{1-2i}{5}e^{it} = \tfrac{1-2i}{5}(\cos t + i \sin t).$$
Now
$$\mathrm{Re}(y_p) = \tfrac{1}{5}\cos t + \tfrac{2}{5}\sin t, \qquad \mathrm{Im}(y_p) = -\tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t,$$
and hence we conclude that the two general solutions are given by
$$\begin{aligned} y_a(t) &= e^{-t}(c_1 \cos t + c_2 \sin t) + \tfrac{1}{5}\cos t + \tfrac{2}{5}\sin t, \\ y_b(t) &= e^{-t}(c_1 \cos t + c_2 \sin t) - \tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t. \end{aligned}$$

5. (a) The characteristic polynomial of the homogeneous equation is given by

$$\lambda^2 + 3\lambda + 2 = 0,$$

whose roots are $\lambda_1 = -1$ and $\lambda_2 = -2$. In particular, $y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$ for some constants c_1, c_2. Since the right hand side of the original ODE has the same form as one of the fundamental solutions, an ansatz $y_p \sim e^{-t}$ will not work (try it!). Instead, use $y_p(t) = Cte^{-t}$; then

$$y_p'(t) = (C - Ct)e^{-t}, \qquad y_p''(t) = (-2C + Ct)e^{-t}.$$

After substituting into the ODE and cancelling the common factor e^{-it}, we have

$$\begin{aligned} -2C + Ct + 3C - 3Ct + 2Ct &= 2 \\ t(C - 3C + 2C) + C &= 2, \end{aligned}$$

so $C = 2$. The general solution is therefore

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + 2te^{-t},$$

for some constants c_1, c_2.

(b) The homogeneous equation is the same as in q.4, and the solution is therefore $y_h(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$. Given that the right hand side of the ODE is $5 \cosh t$, we wish to try an ansatz of the form $y_p(t) = C \cosh t$. When differentiated, this will also produce terms proportional to $\sinh t$; we balance this by using the ansatz $y_p(t) = A \cosh t + B \sinh t$; then

$$y_p'(t) = A \sinh t + B \cosh t, \qquad y_p''(t) = A \cosh t + B \sinh t = y_p(t).$$

Substituting into the ODE gives us

$$\begin{aligned} 3(A \sinh t + B \cosh t) + 2(A \sinh t + B \cosh t) &= 5 \cosh t \\ \cosh t(3A + 2B - 5) + \sinh t(3B + 2A) &= 0. \end{aligned}$$

In particular, $3A + 2B - 5 = 0$ and $3B + 2A = 0$, from which we extract that $A = 3$, $B = -2$. The general solution is therefore

$$y(t) = e^{-t}(c_1 \cos t + c_2 \sin t) + 3 \cosh t - 2 \sinh t,$$

for some constants c_1, c_2.

(c) The homogeneous equation is the same as in q. 5(a), so $y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$. As in the previous question, we are tempted to try an ansatz that contains both $\cosh t$ and $\sinh t$ terms. This will however not work, since $2 \cosh t = e^t + e^{-t}$ contains one of the fundamental solutions. Instead, we look for a particular solution of the form $y_p(t) = Ate^{-t} + Be^t$; then

$$y_p'(t) = e^{-t}(A - At) + Be^t, \qquad y_p''(t) = e^{-t}(-2A + At) + Be^t.$$

Substituting into the ODE gives us

$$\begin{aligned}e^{-t}(-2A + At) + Be^t + 3e^{-t}(A - At) + 3Be^t + 2Ate^{-t} + 2Be^t &= e^t + e^{-t} \\ e^{-t}(-2A + At + 3A - 3At + 2At) + e^t(B + 3B + 2B) &= e^t + e^{-t} \\ Ae^{-t} + 6Be^t &= e^t + e^{-t}.\end{aligned}$$

Hence, $A = 1$ and $B = \frac{1}{6}$. The general solution is therefore given by

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + te^{-t} + \tfrac{1}{6}e^t,$$

for some constants c_1, c_2.

6. This is a linear inhomogeneous equation. We present a slightly different way of finding the solution as usual.
Set $u = x^2 + 1$ and integrate the linear factor by substitution such that

$$\int \frac{2x}{x^2+1}\,\mathrm{d}x = \int \frac{1}{u} \cdot 2x \cdot \frac{\mathrm{d}u}{2x} = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln(x^2+1) + C.$$

The *integrating factor* now becomes

$$e^{\int \frac{2x}{x^2+1}\,\mathrm{d}x} = e^{\ln(x^2+1)} = x^2 + 1.$$

Multiply the original equation by $x^2 + 1$ to find

$$y' \cdot (x^2+1) + 2x \cdot y = 4x(x^2+1),$$

and notice that we can apply the product rule for differentiation *in reverse* on the left hand side:

$$y' \cdot (x^2+1) + 2x \cdot y = (y(x^2+1))' = 4x^3 + 4x.$$

Integrating both sides and dividing by $(x^2 + 1)$ we finally arrive at

$$y = \frac{x^4 + 2x^2}{x^2+1} + \frac{C}{x^2+1}.$$

To determine the constant C, insert $x = 1$, $y = 3$:

$$3 = \frac{1+2}{1+1} + \frac{C}{1+1};$$

thus $C = 3$. The solution to the differential equation is therefore $y = \frac{x^4+2x^2+3}{x^2+1}$.

7. (a) This is a linear inhomogeneous equation, and we solve it using the method presented in the lectures; the method presented in q.4 is however also applicable. Start by finding the general solution to the homogeneous equation

$$y' - y = 0.$$

By separation of variables we deduce

$$\int \frac{\mathrm{d}y}{y} = \int \mathrm{d}x,$$

in other words, $y_h(x) = C_1 e^x$ for some constant C_1.
Next, we find a particular solution to the inhomogeneous equation by variation of constants: substitute $y_p(x) = C(x)e^x$ into the original problem to find that

$$C'(x) = e^{-x}\cos x.$$

The latter expression can be solved using integration by parts twice, namely

$$\begin{aligned}\int e^{-x}\cos x\,\mathrm{d}x &= -e^{-x}\cos x - \int e^{-x}\sin x\,\mathrm{d}x \\ &= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,\mathrm{d}x,\end{aligned}$$

such that

$$C(x) = \int_0^x C'(t)\,\mathrm{d}t = \tfrac{1}{2}(\sin x - \cos x)e^{-x} + C_2.$$

A particular solution is therefore given by $y_p(x) = C(x)e^x = \frac{1}{2}(\sin x - \cos x) + C_2 e^x$.

The general solution is now $y = y_h + y_p = (C_1 + C_2)e^x + \frac{1}{2}(\sin x - \cos x)$; since the initial condition $y(0) = 0$ gives $C_1 + C_2 = \frac{1}{2}$, we conclude that

$$y(x) = \tfrac{1}{2}(\sin x - \cos x + e^x).$$

(b) This is again a linear inhomogeneous equation, and we start by finding the solution of

$$y' + \frac{3y}{x} = 0,$$

as described in the lectures (method presented in q.4 is also applicable). Separating variables and integrating gives

$$\frac{1}{3}\int \frac{\mathrm{d}y}{y} = -\int \frac{\mathrm{d}x}{x},$$

and so the general solution to the homogeneous system is $y_h(x) = C_1 x^{-3}$ for some constant C_1.
To find a particular solution, substitute $y_p(x) = C(x)x^{-3}$ into the original equation to deduce that

$$C'(x) = 5x^4.$$

In other words $C(x) = x^5 + C_2$, and so $y_p(x) = x^2 + C_2 x^{-3}$. The general solution to our system is therefore $y = y_h + y_p = x^2 + (C_1 + C_2)x^{-3}$; since the initial condition $y(1) = 2$ gives $C_1 + C_2 = 1$, we conclude that

$$y(x) = x^2 + x^{-3}.$$

(c) By separation of variables we have

$$\int \frac{\mathrm{d}y}{1+y^2} = \int \frac{\mathrm{d}x}{1+x^2},$$

and so $\arctan y = \arctan x + C_1$ for some constant C_1. We use the formula

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

to write the solution as

$$y = \tan(\arctan x + C_1) = \frac{\tan(\arctan x) + \tan C_1}{1 - \tan(\arctan x) \cdot \tan C_1} = \frac{x + \tan C_1}{1 - x \tan C_1}$$

Denote $C = \tan C_1$, so that C is just another constant;
this leads to the solution $y(x) = \frac{x+C}{1-Cx}$.