LINEAR SYSTEMS AND PROPERTIES OF FUNCTIONS

1. (a) The function f is surjective: For any $y \in \mathbb{R}_{\geq 0}$, we have $f(y) = |y| = y$. However, f is not injective as for example $f(-1) = f(1) = 1$.

   (b) The function g is not surjective, since 1 is not contained in its image. It is nevertheless injective: if $g(a) = g(b)$, then $a + 1 = b + 1$, so $a = b$.

   (c) The function h is bijective. We first show surjectivity. Let $k \in \mathbb{Z}$ be any integer. if $k \geq 0$, then $h(2k) = k$. If $k < 0$, then $h(-2k + 1) = k$. This shows that every element of $\mathbb{Z}$ is in the image of h.

   To prove that h is injective, assume that $h(m) = h(n)$. If this is simply zero, it immediately follows that $m = n = 0$. Suppose therefore that $h(n) = h(m)$ is equal to a positive integer. Then $\frac{m}{2} = \frac{n}{2}$, so $m = n$. If on the other hand $h(m) = h(n)$ is a negative number, m and n must be odd; from $-\frac{m+1}{2} = -\frac{n+1}{2}$ we again deduce that $m = n$.

   Using the above considerations we deduce that the inverse of h is given by

$$h^{-1} : \mathbb{Z} \to \mathbb{N}_0, \quad h^{-1}(k) = \begin{cases} 2k, & \text{if } k \geq 0 \ , \\ -2k - 1, & \text{if } k < 0. \end{cases}$$

2. (a) Comparing the first row with the third yields $z = 1$. Comparing the first row with the second yields $y = 2$. If we now substitute $y = 2$ and $z = 1$ in the first row, we find that $x = 1$. The unique solution of the system is therefore the point $(1, 2, 1) \in \mathbb{R}^3$.

   (b) Subtracting three times the first row from the second, we find that $y = 1$. Substituting this into the first row gives $x = 4$. The point $(4, 1) \in \mathbb{R}^2$ is exactly the intersection of the two lines $\{y = \frac{x}{2} - 1\}$ and $\{y = -\frac{3}{5}x + \frac{17}{5}\}$.

   (c) By elimination, we reduce the system to

$$\left| \begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ & - & 3y & + & 3z & = & 0 \\ & - & 6y & + & 6z & = & 1 \end{array} \right| .$$

   Comparing the last two rows, we see that the system is inconsistent; it has no solution, because $1 = 0$ is never true.

3. (a) We first reduce the system to

$$\left| \begin{array}{ccccccc} x & + & y & - & z & = & -2 \\ & & y & - & 2z & = & -3 \\ & & & & 0 & = & k - 7 \end{array} \right| ,$$

   hence $k = 7$ is a necessary condition for the system to be consistent.

(b) For $k = 7$, the last row simply becomes $0 = 0$, and we may discard it. The remaining system now consists of two equations in three variables; geometrically we are considering two planes. If these equations admit simultaneous solutions, the two planes must intersect in a line. Each point of this line will be a solution, so in other words there are infinitely many solutions.

(c) The line described above can be parametrized as $(1, -3, 0) + t(-1, 2, 1)$, $t \in \mathbb{R}$. To see this, recall that the parametrization of a line is given by $u + tv$, where u denotes a point on the line, and v a vector parallel to it. To find such a vector v, it suffices to take the difference of two points on the line. The suggested parametrization is constructed using $u = (1, -3, 0)$ and $v = (0, -1, 1) - (1, -3, 0)$.

4. Proceed by backward substitution: The last row tells you that $x_4 = 0$. Using this information in the second-to-last row, find the value of x_3. Now proceed inductively: by substituting the value of x_3 and x_4 in the second row you find the value of x_2, and so on.

5. We aim to find a polynomial $f(t) = at^2 + bt + c$ such that $f(-1) = 1$, $f(2) = 3$, $f(3) = 13$. This amounts to solving the linear system

$$\left|\begin{array}{ccccccc} a & - & b & + & c & = & 1 \\ 4a & + & 2b & + & c & = & 3 \\ 9a & + & 3b & + & c & = & 13 \end{array}\right|,$$

which we immediately reduce to

$$\left|\begin{array}{ccccccc} a & - & b & + & c & = & 1 \\ & & 6b & + & -3c & = & -1 \\ & & & & -2c & = & 6 \end{array}\right|.$$

In particular $c = -3$; using the first and second row above, we now easily compute $b = -\frac{5}{3}$ and $a = \frac{7}{3}$. The resulting polynomial is therefore $f(t) = \frac{7}{3}t^2 - \frac{5}{3}t - 3$.

6. Let x denote the number of 20ct coins, y the number of 50ct coins and z the number of 1 Fr. coins. To receive the award, the following two conditions must be satisfied:

$$\left|\begin{array}{ccccccc} x & + & y & + & z & = & 1000 \\ \frac{1}{5}x & + & \frac{1}{2}y & + & z & = & 1000 \end{array}\right|.$$

The solution of the system is $x = -\frac{5}{3}(1000 - z)$, $y = \frac{8}{3}(1000 - z)$. By definition, x needs to be a non-negative integer. Now since $z \leq 1000$ and $x \geq 0$, the only possible solution is $x = 0$, $y = 0$ and $z = 1000$.

7. <u>AB</u>. This is 2×3 times 3×2, which will give us a 2×2 matrix.

$$AB = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 0 \times 3 + (-1) \times 1 + 2 \times 6 & 0 \times (-1) + -1 \times 2 + 2 \times 1 \\ 4 \times 3 + 11 \times 1 + 2 \times 6 & 4 \times (-1) + 11 \times 2 + 2 \times 1 \end{pmatrix}$$

which gives a two by two matrix

$$\begin{pmatrix} 11 & 0 \\ 35 & 20 \end{pmatrix}.$$

In a similar way, we compute the three by three matrix

$$BA = \begin{pmatrix} -4 & -14 & 4 \\ 8 & 21 & 6 \\ 4 & 5 & 14 \end{pmatrix}.$$

Moreover, we see that AB is not equal to AB. In fact, for most matrices, you cannot reverse the order of multiplication and get the same result.