

Mathematics

1. System of linear equations

Consider the equation

$$\begin{pmatrix} 1 & 3 \\ 2 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4-t \end{pmatrix}.$$

- (a) For which values of the parameter $t \in \mathbb{R}$ does the above equation have a unique solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$?
- (b) Determine this unique solution.

2. Eigenvalues and eigenvectors

For the 3×3 -matrix

$$A = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & 0 & 1 \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

determine a diagonal matrix B and a transformation matrix T such that

$$B = T^{-1}AT.$$

3. Complex numbers

- (a) Compute

$$(z-1)(z^2+z+1).$$

- (b) Use your previous result in order to determine all the solutions of

$$z^2 + z + 1 = 0$$

in their polar form $z = r \exp(i\varphi)$ with $0 \leq r \in \mathbb{R}$ and $0 \leq \varphi < 2\pi$.

4. Extrema

Find the global maximum M and the global minimum m of the function

$$f(x) := x^4 + 4x^3 - 20x^2 + 15$$

on the interval $[-6, 4]$ and give the two values $x_{\max}, x_{\min} \in [-6, 4]$, where the global maximum M and the global minimum m is attained.

5. First order differential equation

Let $x \geq 0$. Find the solution $y(x)$ of the differential equation

$$(x^2 + 3x + 2)y' + y^2 = 0$$

satisfying the initial condition

$$y(0) = \frac{1}{1 - \ln(2)}.$$

6. Linear differential equations with constant coefficients

Find the solution $y(x)$ of the differential equation

$$y'' - 2y' - 3y = -10 \cos(x)$$

that is bounded for $x \rightarrow \infty$ and that satisfies $y(0) = 1$.

7. System of linear differential equations

Determine the solutions $x_1(t), x_2(t)$ of the following system of differential equations

$$\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 \\ \dot{x}_2 &= + 5x_2 \end{aligned}$$

that satisfy $x_1(0) = 0, x_2(0) = 1$.