Mathematics

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1. **System of linear equations**

Consider the equation

$$\begin{pmatrix} 1 & 3 \\ 2 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4-t \end{pmatrix}.$$

(a) For which values of the parameter $t \in \mathbb{R}$ does the above equation have a unique solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$?

(b) Determine this unique solution.

Solution.

(a) The equation $A_t x = b_t$ has a unique solution if and only if A_t is invertible. Let

$$A_t := \begin{pmatrix} 1 & 3 \\ 2 & t \end{pmatrix}.$$

Then $\det A_t = t - 6$ and A_t is invertible if and only if $t \neq 6$.

(b) For $t \neq 6$ we have

$$A_t^{-1} = \frac{1}{t-6} \begin{pmatrix} t & -3 \\ -2 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{1}{t-6} \begin{pmatrix} t & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4-t \end{pmatrix} \\ &= \frac{1}{t-6} \begin{pmatrix} 2\,t - 12 \\ 6 - t \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

is the unique solution.

2. **Eigenvalues and eigenvectors**

For the 3×3-matrix

$$A = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & 0 & 1 \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

determine a diagonal matrix B and a transformation matrix T such that

$$B = T^{-1}AT\,.$$

Solution.

We compute the eigenvectors and the eigenvalues of A.

The eigenvalues of the matrix A are the zeros of the polynomial

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda 1) &= \det \begin{pmatrix} \frac{1}{2} - \lambda & 1 & -\frac{3}{2} \\ 1 & -\lambda & 1 \\ -\frac{3}{2} & 1 & \frac{1}{2} - \lambda \end{pmatrix} \\ &= -\lambda\Big(\frac{1}{2} - \lambda\Big)^2 - \frac{3}{2} - \frac{3}{2} + \frac{9}{4}\lambda - \Big(\frac{1}{2} - \lambda\Big) - \Big(\frac{1}{2} - \lambda\Big) \\ &= -\lambda\Big(\frac{1}{4} - \lambda + \lambda^2\Big) - 3 + \frac{9}{4}\lambda - 1 + 2\lambda \\ &= -\lambda^3 + \lambda^2 - \frac{1}{4}\lambda + \frac{9}{4}\lambda + 2\lambda - 4 \\ &= -\lambda^3 + \lambda^2 + 4\lambda - 4 \\ &= (1 - \lambda)(\lambda^2 - 4) = (1 - \lambda)(\lambda - 2)(\lambda + 2)\,. \end{aligned}$$

These are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -2$. We determine an eigenvector v_1 for $\lambda_1 = 1$.

$$(A - 1)v_1 = \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & -1 & 1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{!}{=} 0\,.$$

Hence the entries x, y and z of $v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfy

$$\begin{aligned} -x + 2y - 3z &= 0 \\ x - y + z &= 0\,. \end{aligned}$$

Substituting $x = y - z$ in the first equation we get $y - 2z = 0$. Hence

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} .$$

We determine an eigenvector v_2 for $\lambda_2 = 2$.

$$(A - \lambda_2)v_2 = \begin{pmatrix} -\frac{3}{2} & 1 & -\frac{3}{2} \\ 1 & -2 & 1 \\ -\frac{3}{2} & 1 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{!}{=} 0 .$$

Hence the entries x, y and z of $v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfy

$$\begin{array}{rcrcrcl} -3x & + & 2y & - & 3z & = & 0 \\ x & - & 2y & + & z & = & 0 . \end{array}$$

Substituting $x = 2y - z$ in the first equation we get $y = 0$. Hence

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} .$$

We determine an eigenvector v_3 for $\lambda_3 = -2$.

$$(A - \lambda_2)v_2 = \begin{pmatrix} \frac{5}{2} & 1 & -\frac{3}{2} \\ 1 & 2 & 1 \\ -\frac{3}{2} & 1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{!}{=} 0 .$$

Hence the entries x, y and z of $v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfy

$$\begin{array}{rcrcrcl} 5x & + & 2y & - & 3z & = & 0 \\ x & + & 2y & + & z & = & 0 \\ -3x & + & 2y & + & 5z & = & 0 . \end{array}$$

This is equivalent to

$$\begin{array}{rcrcrcl} 4x & & & - & 4z & = & 0 \\ x & + & 2y & + & z & = & 0 . \end{array}$$

Substituting $x = z$ in the second equation we get $x + y = 0$. Hence

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} .$$

This shows that

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

satisfies

$$B = T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

3. **Complex numbers**

   (a) Compute
   $$(z-1)(z^2+z+1) .$$

   (b) Use your previous result in order to determine all the solutions of
   $$z^2 + z + 1 = 0$$
   in their polar form $z = r\exp(i\varphi)$ with $0 \le r \in \mathbb{R}$ and $0 \le \varphi < 2\pi$.

Solution.

(a) Compute
$$\begin{aligned}(z-1)(z^2+z+1) &= z^3 + z^2 + z - z^2 - z - 1 \\ &= z^3 - 1\end{aligned}$$

(b) The solutions of
$$z^2 + z + 1 = 0$$
are also solutions of $z^3 - 1 = 0$. These can be found using the polar form $z = r\exp(i\varphi)$ with $0 \le r \in \mathbb{R}$ and $0 \le \varphi < 2\pi$. Since
$$z^3 = r^3 \exp(i3\varphi) ,$$
the solutions of $z^3 = r^3\exp(i3\varphi) = 1$ satisfy $r^3 = 1$ and $3\varphi = k2\pi$, $k \in \mathbb{Z}$. The three solutions
$$z_1 = 1, \quad z_2 = \exp\left(\frac{i2\pi}{3}\right), \quad z_3 = \exp\left(\frac{i4\pi}{3}\right)$$

are different. Since $z_1 = 1$ is a solution of $z^3 - 1 = 0$ and not a solution of z^2+z+1, the solutions of $z^2 + z + 1 = 0$ are

$$z_2 = \exp\Big(\frac{i2\pi}{3}\Big) \quad \text{and} \quad z_3 = \exp\Big(\frac{i4\pi}{3}\Big).$$

4. **Extrema**

Find the global maximum M and the global minimum m of the function

$$f(x) := x^4 + 4x^3 - 20x^2 + 15$$

on the interval $[-6, 4]$ and give the two values $x_{\max}, x_{\min} \in [-6, 4]$, where the global maximum M and the global minimum m is attained.

Solution.

We first find the critical points (the local extrema), that is, the points x for which $f'(x) = 0$. The derivative is

$$\begin{aligned} f'(x) &= (x^4 + 4x^3 - 20x^2 + 15)' \\ &= (x^4)' + 4(x^3)' - 20(x^2)' + 15' \\ &= 4x^3 + 4 \cdot 3x^2 - 20 \cdot 2x + 0 \\ &= 4x^3 + 12x^2 - 40x \\ &= 4x(x^2 + 3x - 10) \\ &= 4x(x-2)(x+5). \end{aligned}$$

Hence the local extrema occur at $x_1 = -5$, $x_2 = 0$ and $x_3 = 2$.
To determine whether the f-values at these three points are local maxima or local minima, we look at the second derivative, which is given by

$$\begin{aligned} f''(x) &= (f'(x))' \\ &= (4x^3 + 12x^2 - 40x)' \\ &= 4(x^3)' + 12(x^2)' - 40x' \\ &= 4 \cdot 3x^2 + 12 \cdot 2x - 40 \\ &= 12x^2 + 24x - 40 \\ &= 4(3x^2 + 6x - 10). \end{aligned}$$

Because we have that

$$\begin{aligned} f''(-5) &= 4 \cdot (3 \cdot (-5)^2 + 6 \cdot (-5) - 10) = 4 \cdot (75 - 30 - 10) = 140 > 0, \\ f''(0) &= 4 \cdot (3 \cdot 0^2 + 6 \cdot 0 - 10) = -40 < 0, \\ f''(2) &= 4 \cdot (3 \cdot 2^2 + 6 \cdot 2 - 10) = 4 \cdot (12 + 12 - 10) = 56 > 0, \end{aligned}$$

we have that $x_2 = 0$ corresponds to a local maximum and that $x_1 = -5$ and $x_3 = 2$ correspond to local minima.

To find the global extrema on the interval $[-6, 4]$, we compare the values of f at the points $-5, 0, 2$ with the two endpoints $e_1 = -6$ and $e_2 = 4$ of the interval $[-6, 4]$, over which the global extrema should be identified.
Putting everything together, we have that

$$\begin{aligned}
f(-6) &= (-6)^4 + 4 \cdot (-6)^3 - 20 \cdot (-6)^2 + 15 = 1296 - 864 - 720 + 15 = -273, \\
f(-5) &= (-5)^4 + 4 \cdot (-5)^3 - 20 \cdot (-5)^2 + 15 = 625 - 500 - 500 + 15 = -360, \\
f(0) &= 0^4 + 4 \cdot 0^3 - 20 \cdot 0^2 + 15 = 15, \\
f(2) &= 2^4 + 4 \cdot 2^3 - 20 \cdot 2^2 + 15 = 16 + 32 - 80 + 15 = -17, \\
f(4) &= 4^4 + 4 \cdot 4^3 - 20 \cdot 4^2 + 15 = 256 + 256 - 320 + 15 = 207.
\end{aligned}$$

Hence the global maximum M over the interval $[-6, 4]$ of the function

$$f(x) = x^4 + 4x^3 - 20x^2 + 15$$

is $M = 207$ and the global minimum m is $m = -360$.
These two global extrema occur at the points $x_{\max} = 4$ and $x_{\min} = -5$, because we have that $f(4) = 207$ and that $f(-5) = -360$.

5. **First order differential equation**

Let $x \geq 0$. Find the solution $y(x)$ of the differential equation

$$(x^2 + 3x + 2)y' + y^2 = 0$$

satisfying the initial condition

$$y(0) = \frac{1}{1 - \ln(2)}.$$

Solution.

By separating variables, we get

$$
\begin{aligned}
& (x^2+3x+2)y' + y^2 = 0 \\
\iff & (x^2+3x+2)y' = -y^2 \\
\iff & -y' = \frac{y^2}{x^2+3x+2} \\
\iff & -\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^2+3x+2} \\
\iff & -\frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{x^2+3x+2} \\
\iff & -\int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{x^2+3x+2} \\
\iff & -\int \frac{\mathrm{d}y}{y^2} = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) \mathrm{d}x \\
\iff & \frac{1}{y} = \ln(|x+1|) - \ln(|x+2|) + C \ \text{ for some } C \in \mathbb{R} \\
\iff & y(x) = \frac{1}{\ln(|x+1|) - \ln(|x+2|) + C} \ \text{ for some } C \in \mathbb{R}.
\end{aligned}
$$

The initial condition

$$y(0) = \frac{1}{1-\ln(2)} = \frac{1}{\ln(1)-\ln(2)+C} = \frac{1}{C-\ln(2)},$$

implies that

$$C = 1.$$

Therefore, we get that the solution to the differential equation for $x \geq 0$ is

$$y(x) = \frac{1}{\ln(x+1) - \ln(x+2) + 1}.$$

6. **Linear differential equations with constant coefficients**

Find the solution $y(x)$ of the differential equation

$$y'' - 2y' - 3y = -10\cos(x)$$

that is bounded for $x \to \infty$ and that satisfies $y(0) = 1$.

Solution.

We must solve the linear differential equation with constant coefficients

$$y'' - 2y' - 3y = -10\cos(x)$$

We first solve the corresponding homogeneous differential equation

$$y'' - 2y' - 3y = 0\,.$$

The characteristic polynomial $p(\lambda)$ of this homogeneous differential equation is

$$\begin{aligned} p(\lambda) &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda+1)(\lambda-3)\,. \end{aligned}$$

Hence the zeros are $\lambda_1 = -1$ and $\lambda_2 = 3$. The general solution $y_{\text{hom}}(x)$ of the homogeneous differential equation is therefore given by

$$y_{\text{hom}}(x) = C_1 e^{-x} + C_2 e^{3x} \quad \text{for some } C_1, C_2 \in \mathbb{R}\ .$$

For the particular solution $y_{\text{part}}(x)$ of the inhomogeneous differential equation

$$y'' - 2y' - 3y = -10\cos(x)$$

we make the "ansatz" $y_{\text{part}}(x) = A\cos(x) + B\sin(x)$

$$\begin{aligned} y_{\text{part}}(x) &= A\cos(x) + B\sin(x) \\ y'_{\text{part}}(x) &= B\cos(x) - A\sin(x) \\ y''_{\text{part}}(x) &= -A\cos(x) - B\sin(x)\,. \end{aligned}$$

By setting this "ansatz" into the differential equation, we get

$$\begin{aligned} y'' - 2y' - 3y &= -A\cos(x) - B\sin(x) - 2\big(B\cos(x) - A\sin(x)\big) \\ &\quad - 3\big(A\cos(x) + B\sin(x)\big) \\ &= (-4A - 2B)\cos(x) + (2A - 4B)\sin(x) \\ &\overset{!}{=} -10\cos(x)\,. \end{aligned}$$

Comparing the coefficients we get

$$\begin{aligned} -4A - 2B &= -10 \\ 2A - 4B &= 0\,. \end{aligned}$$

Hence $A = 2B$ and $4A + 2B = 5A = 10$ and $A = 2$, $B = 1$. We obtain

$$y_{\text{part}}(x) = 2\cos(x) + \sin(x)\,.$$

The general solution $y(x)$ to the inhomogeneous differential equation is therefore given by

$$y(x) = y_{\text{hom}}(x) + y_{\text{part}}(x) = C_1 e^{-x} + C_2 e^{3x} + 2\cos(x) + \sin(x)$$

for some $C_1, C_2 \in \mathbb{R}$. Since $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty} e^{3x} = \infty$, the solution is bounded if and only if $C_2 = 0$. The condition $y(0) = 1$ yields

$$y(0) = C_1 + 2\cos(0) + \sin(0) = C_1 + 2 \overset{!}{=} 1\,,$$

hence $C_1 = -1$. The solution of the differential equation is

$$y(x) = -e^{-x} + 2\cos(x) + \sin(x)\,.$$

7. **System of linear differential equations**

Determine the solutions $x_1(t)$, $x_2(t)$ of the following system of differential equations

$$\begin{array}{rcrcr} \dot{x}_1 & = & 2x_1 & + & 3x_2 \\ \dot{x}_2 & = & & & 5x_2 \end{array}$$

that satisfy $x_1(0) = 0$, $x_2(0) = 1$.

Solution.

The system can be written as $\dot{x} = Ax$ with

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 5$. The eigenvectors v_1, v_2 to the eigenvalues λ_1, λ_2 respectively are

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

With the transformation matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and its inverse } T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

we get the diagonal matrix

$$D = T^{-1}AT = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$

The solution of the system

$$\begin{aligned} \dot{y}_1 &= 2\,y_1 \\ \dot{y}_2 &= 5\,y_2 \end{aligned}$$

is $y_1 = C_1 e^{2t}$ and $y_2 = C_2 e^{5t}$. With $x = Ty$, i.e. $x_1 = y_1 + y_2$, $x_2 = y_2$, we get

$$\begin{aligned} x_1(t) &= C_1 e^{2t} + C_2 e^{5t} \\ x_2(t) &= C_2 e^{5t}\,. \end{aligned}$$

We use the initial condition $x_1(0) = 0$, $x_2(0) = 1$ to determine the constants C_1 and C_2. With

$$\begin{aligned} x_1(0) &= C_1 + C_2 = 0 \\ x_2(0) &= C_2 = 1\,, \end{aligned}$$

we get $C_2 = 1$ and $C_1 = -C_2 = -1$. Hence

$$\begin{aligned} x_1(t) &= -e^{2t} + e^{5t} \\ x_2(t) &= e^{5t}\,. \end{aligned}$$