1. **Area enclosed by a graph**

   Consider the real-valued function $f(x) = (\ln x)^2 - 1$.

   (a) Find the zeros of f and sketch its graph, clearly indicating these zeros.

   (b) Use integration by parts to show the following two identities:

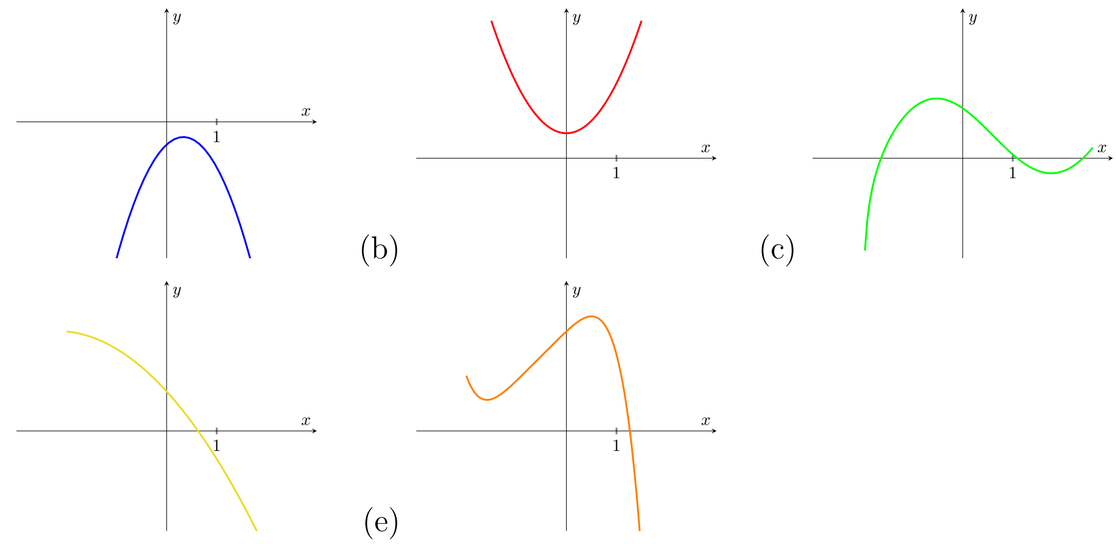
   $$\int \ln x \, \mathrm{d}x = x \cdot \ln x - x + c_1, \qquad \int (\ln x)^2 \, \mathrm{d}x = x \cdot (\ln x)^2 - 2x \cdot \ln x + 2x + c_2,$$

   where $c_1, c_2 \in \mathbb{R}$.

   (c) Compute the area enclosed by the graph of f, the x-axis, and the two lines $x = 1$, $x = 5$.

2. **Evaluation of a function and its derivatives**

   Which of the following graph(s) has the properties: $f'(0) > 0$, $f'(1) < 0$ and $f''(x) < 0$ for all values of x? Justify *all* of your answers.



(a) (b) (c) (d) (e)

3. **Complex numbers**

   (a) Sketch the following subset of $\mathbb{C}$:

   $$\{z \in \mathbb{C} \mid |z - i| > 1 \text{ and } |z| < 2\}.$$

   (b) Write the expression $\frac{3+i}{2-i}$ in the form $x + iy$, with $x, y \in \mathbb{R}$.

   (c) Write $1 + i$ in polar form $re^{i\varphi}$.

4. **Linear differential equations with constant coefficients**

Find the general solutions to the following two differential equations.

(a) $y'' + 2y' + 2y = 0$, subject to $y(\pi) = 0$ and $y'(\pi) = 2e^{-\pi}$,

(b) $y'' = 1 + t^2$.

5. **System of linear equations**

Consider the equation

$$\begin{pmatrix} 1 & 0 & t \\ 5 & -1 & 10 \\ -1 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}.$$

(a) For which values of the parameter $t \in \mathbb{R}$ does the above equation have a unique solution $x \in \mathbb{R}^3$?

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

(b) Determine the solution for the case $t = -1$.

6. **Eigenvalues and eigenvectors**

Let A be the 2×2-matrix

$$A = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}.$$

(a) Find an eigenvector for each eigenvalue of the matrix A.

(b) Consider the mapping

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f(x) := Ax\,. \end{array}$$

Compute the image $f(x)$ of

$$x := \begin{pmatrix} \sqrt{3} - 1 \\ \sqrt{3} + 1 \end{pmatrix},$$

where x is written in the standard basis. Give a geometric interpretation of the result.

7. **System of linear differential equations**

Determine the functions $x_1(t)$ and $x_2(t)$ that solve the system of linear differential equations

$$\begin{aligned} \dot{x_1} &= 1x_1 + 2x_2, \\ \dot{x_2} &= 3x_1 + 2x_2, \end{aligned}$$

subject to the initial conditions $x_1(0) = 0$ and $x_2(0) = 5$.