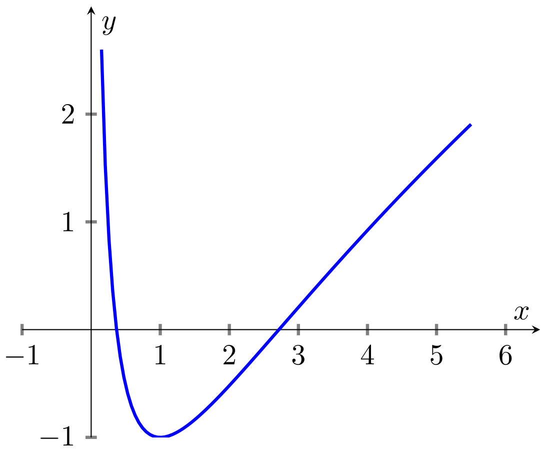
1. **Area enclosed by a graph**

(a) The solutions of the equation $f(x) = (\ln x)^2 - 1 = 0$ are implicitly given by $\ln x_1 = -1$ and $\ln x_2 = 1$. In other words, the zeros of f are $x_1 = e^{-1} = \frac{1}{e}$ and $x_2 = e^1 = e$.

Below is a sketch of the graph of f:



(b) For the first integral, set $u'(x) = 1$, $v(x) = \ln x$ and directly calculate:

$$\begin{aligned}\int 1 \cdot \ln x \, \mathrm{d}x &= x \cdot \ln x - \int x \cdot \frac{1}{x} \, \mathrm{d}x \\ &= x \cdot \ln x - \int 1 \, \mathrm{d}x = x \cdot \ln x - x + c_1,\end{aligned}$$

for some $c_1 \in \mathbb{R}$. For the second integral, proceed with the choice $u'(x) = 1$, $v(x) = (\ln x)^2$. Applying the chain rule we then first compute

$$v'(x) = 2 \ln x \cdot \tfrac{1}{x} = \tfrac{2 \ln x}{x},$$

and so

$$\begin{aligned}\int 1 \cdot (\ln x)^2 \, \mathrm{d}x &= x \cdot (\ln x)^2 - \int x \cdot \frac{2 \ln x}{x} \, \mathrm{d}x \\ &= x \cdot (\ln x)^2 - 2 \int \ln x \, \mathrm{d}x \\ &= x \cdot (\ln x)^2 - 2(x \cdot \ln x - x) + c_2,\end{aligned}$$

for some $c_2 \in \mathbb{R}$.

(c) Using the above results from (b), we find an expression for the indefinite integral of f:

$$\begin{aligned}\int f(x) \, \mathrm{d}x = \int ((\ln x)^2 - 1) \, \mathrm{d}x &= x \cdot (\ln x)^2 - 2(x \cdot \ln x - x) - x + c_3 \\ &= x \cdot (\ln x)^2 - 2x \cdot \ln x + x + c_3,\end{aligned}$$

for some $c_3 \in \mathbb{R}$. Since the required area lies both below and above the x-axis, we split the integral accordingly:

$$A_1 = \int_1^e f(x)\,\mathrm{d}x = -1, \qquad A_2 = \int_e^5 f(x)\,\mathrm{d}x = 5(\ln 5)^2 - 10\ln 5 + 5$$

The area of the specified region is thus $A = |A_1| + |A_2| = 5(\ln 5)^2 - 10\ln 5 + 6$.

2. **Evaluation of a function and its derivatives**

(a) The graph has all the listed properties.

(b) This graph has none of the listed properties: $f'(0) = 0$, $f'(1) > 0$, and $f''(x) > 0$ for all values of x.

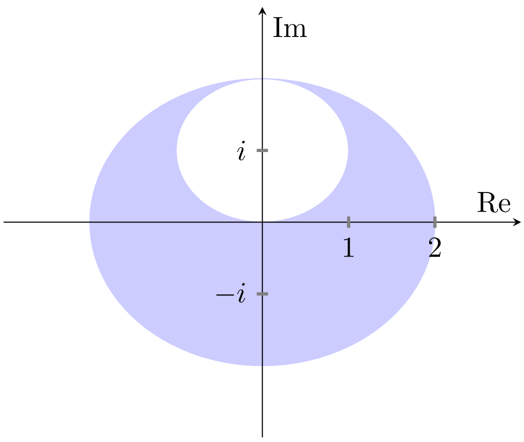
(c) Here $f'(0) < 0$, $f'(1) < 0$, and $f''(x)$ changes sign along the graph.

(d) We have $f'(0) < 0$, $f'(1) < 0$ and finally $f''(x) < 0$ for all values of x since the graph becomes steeper and steeper.

(e) One property is missing since $f'(0) > 0$, $f'(1) < 0$, but $f''(x)$ changes sign along the graph.

We conclude that only the graph in (a) has all the required properties.

3. **Complex numbers**



(a) The shaded region shows the required set. Notice that both circles (centred at $(0, i)$ and the origin) are *open*.

(b) We have

$$\frac{3+i}{2-i} = \frac{(3+i)(2+i)}{(2-i)(2+i)} = \frac{5+5i}{5} = 1+i.$$

(c) For the polar form we get $r = \sqrt{(1+i)(1-i)} = \sqrt{2}$ and $\tan(\varphi) = 1$. Hence, since $1+i$ is in the first quadrant, $\varphi = \frac{\pi}{4}$ and

$$1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}} .$$

4. **Linear differential equations with constant coefficients**

(a) The corresponding characteristic polynomial is given by $\lambda^2 + 2\lambda + 2 = 0$, whose roots are $\lambda_{1,2} = -1 \pm i$. As seen in the lectures, the general solution is therefore

$$y(x) = e^{-x}(A\cos x + B\sin x),$$

for some $A, B \in \mathbb{R}$. In order to apply the initial conditions, we first compute

$$y'(x) = -e^{-x}(A\cos x + B\sin x) + e^{-x}(-A\sin x + B\cos x);$$

then $y(\pi) = 0$ implies that $A = 0$ and $y'(\pi) = 2e^{-\pi}$ implies $B = -2$. We conclude that the final solution is

$$y(x) = -2e^{-x} \cdot \sin x.$$

(b) The solution to the corresponding homogeneous equation $y' = 0$ is

$$y_h = c_1 + c_2 x,$$

for some $c_1, c_2 \in \mathbb{R}$. To find a particular solution, notice that using a complete second-order polynomial as a (naive) ansatz will not work: both constants *and* linear terms are solutions of the homogeneous equation! We thus have to multiply this first guess $y_p^{naive}(x) = A + Bt + Ct^2$ by t^2, to ensure none of its terms are solutions of $y'' = 0$. Then $y_p(x) = At^2 + Bt^3 + Ct^4$, and we compute

$$y_p'(x) = 2At + 3Bt^2 + 4Ct^3, \qquad y_p''(x) = 2A + 6Bt + 12Ct^2.$$

After substituting into the ODE we have

$$(2A + 6Bt + 12Ct^2 = 1 + t^2) \quad \text{or equivalently} \quad (2A-1) + 6Bt + (12C-1)t^2 = 0.$$

In particular, $A = \frac{1}{2}$, $B = 0$ and $C = \frac{1}{12}$. The general solution is therefore

$$y(x) = y_h(x) + y_p(x) = c_1 + c_2 t + \tfrac{1}{2}t^2 + \tfrac{1}{12}t^4.$$

Of course, we could also have found the solution directly by integrating the ODE twice.

5. **System of linear equations**

(a) The equation $A_t x = b_t$ has a unique solution if and only if A_t is invertible. Let

$$A_t := \begin{pmatrix} 1 & 0 & t \\ 5 & -1 & 10 \\ -1 & 1 & 6 \end{pmatrix}.$$

Then $\det A_t = 4t - 16$ and A_t is invertible if and only if $t \neq 4$.

(b) For $t = -1$ we have

$$A := \begin{pmatrix} 1 & 0 & -1 \\ 5 & -1 & 10 \\ -1 & 1 & 6 \end{pmatrix}.$$

We solve the following system of equations.

$$\left| \begin{array}{rcrcrcr} x_1 & & & - & x_3 & = & 3 \\ 5x_1 & - & x_2 & + & 10x_3 & = & 0 \\ -x_1 & + & x_2 & + & 6x_3 & = & 4 \end{array} \right|.$$

This is equivalent to

$$\left| \begin{array}{rcrcrcr} x_1 & & & - & x_3 & = & 3 \\ & - & x_2 & + & 15x_3 & = & -15 \\ & & x_2 & + & 5x_3 & = & 7 \end{array} \right|.$$

and to

$$\left| \begin{array}{rcrcrcr} x_1 & & & - & x_3 & = & 3 \\ & & & & 20x_3 & = & -8 \\ & & x_2 & + & 5x_3 & = & 7 \end{array} \right|.$$

Hence $x_3 = -\frac{2}{5}$, $x_2 = 9$ and $x_1 = \frac{13}{5}$ and the solution is

$$x = \begin{pmatrix} \frac{13}{5} \\ 9 \\ -\frac{2}{5} \end{pmatrix}.$$

It is also possible to determine the solution using the inverse of A. We get $\det(A) = -20$ and

$$A^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{1}{20} & \frac{1}{20} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{5} & \frac{1}{20} & \frac{1}{20} \end{pmatrix}$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{1}{20} & \frac{1}{20} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{5} & \frac{1}{20} & \frac{1}{20} \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{13}{5} \\ 9 \\ -\frac{2}{5} \end{pmatrix}$$

is the unique solution.

6. **Eigenvalues and eigenvectors**

(a) We compute the eigenvectors and the eigenvalues of A.
The eigenvalues are determined by finding the zeros of the characteristic polynomial p_A:

$$
\begin{aligned}
p_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} \frac{3}{4} - \lambda & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} - \lambda \end{pmatrix} \\
&= \Big(\frac{3}{4} - \lambda\Big)\Big(\frac{1}{4} - \lambda\Big) - \frac{3}{16} \\
&= \lambda^2 - \lambda + \frac{3}{16} - \frac{3}{16} \\
&= \lambda^2 - \lambda \\
&= \lambda(\lambda - 1)\,.
\end{aligned}
$$

These are $\lambda_1 = 1$ and $\lambda_2 = 0$. We determine an eigenvector v_1 for $\lambda_1 = 1$.

$$
(A - I)v_1 = \begin{pmatrix} -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{!}{=} 0\,.
$$

Hence the coordinates x, y of $v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfy

$$
\begin{array}{rcrcl}
-\frac{1}{4}\,x & + & \frac{\sqrt{3}}{4}\,y & = & 0 \\
\frac{\sqrt{3}}{4}\,x & - & \frac{3}{4}\,y & = & 0\,.
\end{array}
$$

This system is equivalent to $\sqrt{3}\,x - 3\,y = 0$, and so

$$
v_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}\,.
$$

An eigenvector for $\lambda_2 = 0$ can be determined using the same procedure:

$$
A\,v_2 = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{!}{=} 0,
$$

thus the coordinates x, y of $v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfy

$$
\begin{array}{rcrcl}
\frac{3}{4}\,x & + & \frac{\sqrt{3}}{4}\,y & = & 0 \\
\frac{\sqrt{3}}{4}\,x & + & \frac{1}{4}\,y & = & 0\,.
\end{array}
$$

This system is equivalent to $\sqrt{3}\,x + y = 0$, and we conclude that

$$v_2 = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}.$$

(b) The image $f(x)$ is given by

$$\begin{aligned} f(x) = Ax &= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \sqrt{3}-1 \\ \sqrt{3}+1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3\sqrt{3}}{4} - \frac{3}{4} + \frac{3}{4} + \frac{\sqrt{3}}{4} \\ \frac{3}{4} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}. \end{aligned}$$

Since $x = v_1 + v_2$, we have $f(x) = f(v_1) + f(v_2) = v_1$. The mapping f describes a projection on the subspace of $\mathbb{R}^2$ that is spanned by v_1, i.e., on the line

$$L_{v_1} := \{\mu\, v_1 \mid \mu \in \mathbb{R}\}.$$

Remark: The line $L_{v_2} := \{\nu\, v_2 \mid \nu \in \mathbb{R}\}$ is mapped onto 0.

7. **System of linear differential equations**

We write the system

$$\begin{aligned} \dot{x_1} &= 1x_1 + 2x_2, \\ \dot{x_2} &= 3x_1 + 2x_2, \end{aligned}$$

in matrix form $\dot{\vec{x}} = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad \vec{x} = \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} p_A(\lambda) &= (1-\lambda)(2-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda+1)(\lambda-4), \end{aligned}$$

and so the two eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 4$. Two corresponding eigenvectors are then $\vec{v}_1 = (-1, 1)^\intercal$ and $\vec{v}_2 = (2, 3)^\intercal$, respectively. Thus, in an eigenbasis, A has the representation

$$D = T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, \text{ where } \quad T = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Now, the solution of the system

$$\dot{y}_1 = -y_1$$
$$\dot{y}_2 = 4y_2$$

is given by $y_1(t) = C_1 e^{-t}$ and $y_2(t) = C_2 e^{4t}$ for some constants $C_1, C_2 \in \mathbb{R}$. In particular

$$x_1 = -y_1 + 2y_2 = -C_1 e^{-t} + 2C_2 e^{4t}$$
$$x_2 = y_1 + 3y_2 = C_1 e^{-t} + 3C_2 e^{4t}.$$

Subject to the initial conditions $x_1(0) = 0$ and $x_2(0) = 5$, we have the equations

$$0 = -C_1 + 2C_2$$
$$5 = C_1 + 3C_2,$$

from which we easily determine that $C_1 = 2$ and $C_2 = 1$. The final solution is therefore

$$x_1(t) = -2e^{-t} + 2e^{4t}$$
$$x_2(t) = 2e^{-t} + 3e^{4t}.$$

Alternative method:
In case the matrix A has two real and distinct eigenvalues λ_1 and λ_2 as above, the general solution $\vec{x} = (x_1, x_2)^\intercal$ to any such system is given by

$$\vec{x} = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t},$$

where $\vec{v}_1, \vec{v}_2$ are two corresponding eigenvectors and the constants $C_1, C_2 \in \mathbb{R}$ are determined by the initial conditions (see the solutions of Problem Set 11). In other words, to solve the system it would also suffice to find the eigenvalues and corresponding eigenvectors, then use the general formula and apply the initial conditions.