Mathematics

Cornelia Busch

D-ARCH

September 19, 2023

Mathematics

- Lecturer: PD Dr. Cornelia Busch

  HG G 34.2
  cornelia.busch@math.ethz.ch
  +41 44 632 82 69

- Assistant: Victor Jaeck

  HG J 16.1
  victor.jaeck@math.ethz.ch
  +41 44 632 31 36

Mathematics

- Webpage: Moodle
- Bibliography:

Serge Lang, *A First Course in Calculus*, 5th edition, Springer New York, 1986 (Online resource)

Serge Lang, *Introduction to Linear Algebra* 2nd edition, Springer New York, 1986 (Online resource)

George B. Thomas, Maurice D. Weir, Joel Hass, *Thomas' Calculus : Early Transcendentals : Single Variable*, 13th ed., 2014, Pearson.

George B. Thomas, Maurice D. Weir, Joel Hass, *Thomas' Calculus : Multivariable*, 13th ed., 2014, Pearson.

Lecture

Lecture	Monday	16:15 – 18:00	HG G 26.3
and exercise class	Tuesday	12:15 – 14:00	HG G 26.3
Special offer	Friday	13:15 – 14:00 starting Oct. 8. every odd week (except for the first one).	HG F 26.5

In the weather forecast

This week we loose 4 minutes of daylight every day.

Do you know any mathematical concept that is related to this statement?

In the weather forecast

This week we loose 4 minutes of daylight every day.

We consider a function “daylight”, the function from the days in the calendar to the time interval from 0 minutes to 24 hours. $\rightarrow$ A function . . .

We have a statement about the change of the “daylight” function: At the mid of September it decreases by 4 minutes every day. $\rightarrow$ . . . and its derivative.

Last year a student answered: . . . a differential equation!

Functions

A *function* f from a set A to a set B is a rule that defines for every $x \in A$ a unique $y = f(x) \in B$. We write

$$\begin{aligned} f: \quad A &\rightarrow B \\ x &\mapsto y = f(x)\,. \end{aligned}$$

The derivative

Let

$$\begin{aligned} f: \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) \end{aligned}$$

be a function from $\mathbb{R}$ to $\mathbb{R}$. The *derivative* of f is defined to be the limit

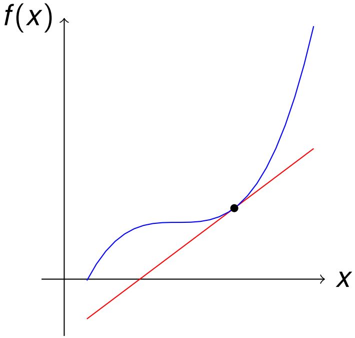
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =: f'(x)$$

Another notation for f' is

$$f'(x) = \frac{d}{dx} f(x) \, .$$

The derivative

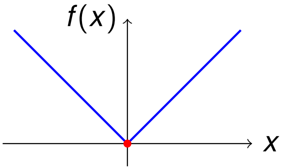
The derivative of a function f in x_0 is the slope of the tangent on the graph of f in the point $(x_0, f(x_0))$.



The derivative measures how the value of the function changes in the neighbourhood of x.

The derivative

The derivative is not defined for each function f or for every point $(x, f(x))$.
Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := |x|$.



Then the derivative is -1 for $x < 0$ and 1 for $0 < x$, but the derivative is not defined in $x = 0$.

Properties

Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be functions and $a, b \in \mathbb{R}$. We assume that the derivatives of f, g and h are defined.

sum	$(f+g)' = f' + g'$
constant factor	$(\lambda f)' = \lambda f'$

Example: Sum and scalar multiplication.

We compute the derivative of

$$f(x) + \lambda g(x)$$

with

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= \sin(x) \text{ and} \\ \lambda &= 3. \end{aligned}$$

$$\left(x^2 + 3\sin(x)\right)' = ?$$

Example: Sum and scalar multiplication.

$$
\begin{aligned}
\left(x^2 + 3\sin(x)\right)' &= \left(x^2\right)' + \left(3\sin(x)\right)' \\
&= \left(x^2\right)' + 3\left(\sin(x)\right)' \\
&= 2x + 3\cos(x)
\end{aligned}
$$

Properties

product	$(fg)' = f'g + fg'$
quotient	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Example: Product rule

We compute the derivative

$$(f \cdot g)' = f'g + fg'$$

for

- $f(x) = x^2$, $g(x) = \sin(x)$
- $f(x) = \cos(x)$, $g(x) = e^x$

Example: Product rule

$$\begin{aligned}\left(x^2\sin(x)\right)' &= \left(x^2\right)'\sin(x) + x^2\left(\sin(x)\right)' \\ &= 2x\sin(x) + x^2\cos(x)\end{aligned}$$

$$\begin{aligned}\left(\cos(x)\cdot e^x\right)' &= \left(\cos(x)\right)'\cdot e^x + \cos(x)\cdot\left(e^x\right)' \\ &= -\sin(x)\cdot e^x + \cos(x)\cdot e^x \\ &= \left(\cos(x) - \sin(x)\right)e^x\end{aligned}$$

Example: Quotient rule

We compute the derivative

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

for

- $f(x) = x^2$, $g(x) = \cos(x)$
- $f(x) = \sin(x)$, $g(x) = \cos(x)$

Example: Quotient rule

$$\left(\frac{x^2}{\cos(x)}\right)' = \frac{(x^2)'\cos(x) - x^2(\cos(x))'}{(\cos(x))^2} = \frac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)}$$
$$= \frac{2x}{\cos(x)} + \frac{x^2}{\cos(x)}\tan(x)$$

$$\left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{(\sin(x))' \cdot \cos(x) - \sin(x) \cdot (\cos(x))'}{(\cos(x))^2}$$
$$= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$
$$= \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

Properties

chain rule	$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$	$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$
inverse	$g'(y) = \frac{1}{f'(g(y))}$	$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$

Example: Chain rule

We compute the derivative of the composition $(f \circ g)(x) = f\big(g(x)\big)$ of two functions

$$(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$$

for $f(x) = \sin(x)$, $g(x) = x^2$. In this case

$$(f \circ g)(x) = f\big(g(x)\big) = \sin(x^2)\,.$$

Exchanging the order of the composition, we get a different function:

$$(g \circ f)(x) = g\big(f(x)\big) = \big(\sin(x)\big)^2 = \sin^2(x)\,.$$

Example: Chain rule

$$
\begin{aligned}
(f \circ g)'(x) &= \left(\sin(x^2)\right)' = \sin'(x^2) \cdot (x^2)' \\
&= \cos(x^2) \cdot (2x) \\
&= 2x\cos(x^2)
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)'(x) &= \left(\sin^2(x)\right)' \\
&= 2\sin(x)\left(\sin(x)\right)' \\
&= 2\sin(x)\cos(x)
\end{aligned}
$$

There is no exercise class this Friday!

Have a good start!