Mathematics

Cornelia Busch

D-ARCH

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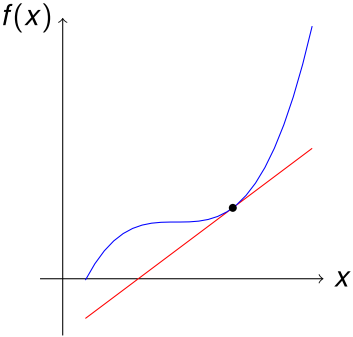
Last time?

- Functions and their derivatives.

The derivative

The derivative of a function f in x_0 is the slope of the tangent on the graph of f in the point $(x_0, f(x_0))$.



The derivative measures how the value of the function changes in the neighbourhood of x.

Properties

Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be functions and $\lambda \in \mathbb{R}$. We assume that the derivatives of f, g and h are defined.

sum	$(f+g)' = f' + g'$
constant factor	$(\lambda f)' = \lambda f'$
product	$(fg)' = f'g + fg'$
quotient	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Properties

Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be functions and $\lambda \in \mathbb{R}$. We assume that the derivatives of f, g and h are defined.

chain rule	$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$	$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$
inverse	$g'(y) = \frac{1}{f'(g(y))}$	$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$

The derivative

Let

$$\begin{aligned} f : \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) \end{aligned}$$

be a function from $\mathbb{R}$ to $\mathbb{R}$. The *derivative* of f is defined to be the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =: f'(x)$$

Another notation for f' is

$$f'(x) = \frac{d}{dx} f(x) \, .$$

Today

- Some definitions about functions.
- Extrema

Functions

“Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes.”

Johann Wolfgang von Goethe

(Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.)

Functions

A *function* f from a set A to a set B is a rule that defines for every $x \in A$ a unique $y = f(x) \in B$. We write

$$\begin{aligned} f : \quad A &\rightarrow B \\ x &\mapsto y = f(x)\,. \end{aligned}$$

We call

- $A = \mathrm{dom}(f)$ the *domain* of f,
- B the *codomain* or *range* of f.
- $\mathrm{im}(f) = \{y \in B \mid \exists x \in A \text{ with } f(x) = y\}$ the *image* of f
- $\mathrm{graph}(f) = \{(x, y) \in A \times B \mid y = f(x)\}$ the *graph* of f.

Functions

$$f_1 : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f_1(x) := x^2 .$$

The function is well-defined since $\forall x \in \mathbb{R}$, $\exists!\, y \in \mathbb{R}$ with $y = x^2$.

- The domain of f_1 is $\mathrm{dom}(f_1) = \mathbb{R}$.
- The range of f_1 is $\mathbb{R}$.
- The image of f_1 is

$$\mathrm{im}(f_1) = \left\{ y \in \mathbb{R} \mid \exists x \in \mathbb{R} \text{ with } y = f_1(x) = x^2 \right\} = \mathbb{R}^{\geqslant 0},$$

  where $\mathbb{R}^{\geqslant 0} = \{y \in \mathbb{R} \mid y \geqslant 0\}$.
- The graph of f_1

$$\mathrm{graph}(f_1) = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 \right\},$$

  is a parabola.

Functions

$$\begin{aligned} f_2 : \quad \mathbb{R} &\rightarrow \mathbb{R}^{\geqslant 0} \\ x &\mapsto f_2(x) := x^2 . \end{aligned}$$

The function is well-defined since $\forall x \in \mathbb{R}, \exists!\, y \in \mathbb{R}^{\geqslant 0}$ with $y = x^2$.

- The domain of f_2 is $\mathrm{dom}(f_2) = \mathbb{R}$.
- The range of f_2 is $\mathbb{R}^{\geqslant 0}$.
- The image of f_2 is
$$\mathrm{im}(f_2) = \mathbb{R}^{\geqslant 0} .$$
- The graph of f_2
$$\mathrm{graph}(f_2) = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^{\geqslant 0} \mid y = x^2\right\} ,$$
is a parabola.

Functions

What is the difference between the following two functions?

$$\begin{aligned} f_1 : \quad \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f_1(x) := \cos(x) \end{aligned}$$

$$\begin{aligned} f_2 : \quad \mathbb{R} &\to [-1, 1] \\ x &\mapsto f_2(x) := \cos(x) \end{aligned}$$

Both functions are well-defined since the image of the cosine is the interval $[-1, 1]$.

Functions

- The domain of f_i, $i = 1, 2$, is $\text{dom}(f_i) = \mathbb{R}$.
- The range of f_1 is $\mathbb{R}$ and the range of f_2 is $[-1, 1]$.
- The image of f_i, $i = 1, 2$, is
$$\text{im}(f_i) = [-1, 1]$$
- The graph of f_1 is
$$\text{graph}(f_1) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = f_1(x)\} .$$
The graph of f_2 is
$$\text{graph}(f_2) = \{(x, y) \in \mathbb{R} \times [-1, 1] \mid y = f_1(x)\} .$$

Continuous functions

Let $f : \mathrm{dom}(f) \to \mathbb{R}$ be a real function, and $I \subseteq \mathrm{dom}(f)$ an open interval.
If $\xi \in I$, then the function f is continuous in ξ if and only if

$$\lim_{x \to \xi} f(x) = f(\xi)\,.$$

Some examples of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$: Any polynomial of degree $n \in \mathbb{N}$:

$$f(x) := a_n x^n + \ldots + a_1 x + a_0$$

where $a_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $a_n \neq 0$. The cosine and the sine function:

$$f(x) := \cos(x), \quad f(x) := \sin(x)$$

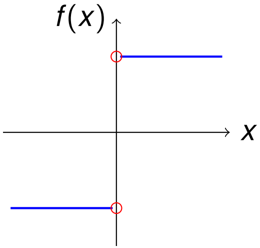
The exponential function and the logarithm:

$$f(x) := e^x, \quad f(x) := \ln(x)$$

Continuous functions

We define

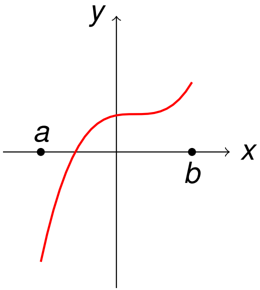
$$\begin{aligned} f : \quad \mathbb{R} \setminus \{0\} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) := \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0\,. \end{cases} \end{aligned}$$



There doesn't exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = f(x)$ for all $x \in \text{dom}(f)$.

Intermediate value theorem

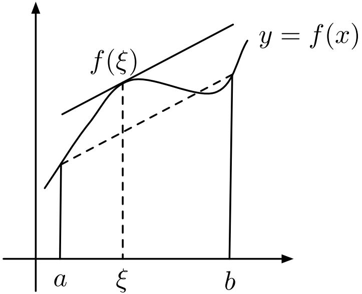
Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0$ and $f(b) > 0$ (resp. $f(a) > 0$ and $f(b) < 0$). Then $p \in (a, b)$ exists with $f(p) = 0$.



Mean value theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable in the inner of $[a, b]$ (i.e. in (a, b)). Then there is a point $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \quad \text{resp.} \quad f(b) - f(a) = f'(\xi)(b - a)\,.$$



Extrema

Let $U \subseteq \mathbb{R}$ be a subset of the real numbers, $f : U \longrightarrow \mathbb{R}$ a function and $x_0 \in U$. Then the following hold.

- f has a local minimum in $x_0 \in U$ if there is an interval $I = (a, b)$ with $x_0 \in I$ and $f(x_0) \leqslant f(x)$ for all $x \in I \cap U$.
- f has a global minimum in $x_0 \in U$ if $f(x_0) \leqslant f(x)$ for all $x \in U$.
- f has a local maximum in $x_0 \in U$ if there is an interval $I = (a, b)$ with $x_0 \in I$ and $f(x_0) \geqslant f(x)$ for all $x \in I \cap U$.
- f has a global maximum in $x_0 \in U$ if $f(x_0) \geqslant f(x)$ for all $x \in U$.

Extrema

Let

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

be a function that is differentiable in $x_0 \in \mathbb{R}$, i.e. the derivative $\frac{d}{dx} f(x_0)$ exists. If

$$\frac{d}{dx} f(x_0) = f'(x_0) = 0$$

and the derivative f' is differentiable in x_0 then

$$\begin{cases} \frac{d^2}{dx^2} f(x_0) = f''(x_0) > 0 & \Rightarrow f(x_0) \text{ is a local minimum of } f \\ \frac{d^2}{dx^2} f(x_0) = f''(x_0) < 0 & \Rightarrow f(x_0) \text{ is a local maximum of } f \end{cases}$$

attained in x_0. No general statement is possible for $f''(x_0) = 0$.

Extrema

Determine the extrema of the function

$$\begin{aligned} f: \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) := x^2 \end{aligned}$$

and of the function

$$\begin{aligned} g: \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) := -x^2 \end{aligned}$$

Extrema

The derivative of the function

$$\begin{array}{rrcl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) := x^2 \end{array}$$

is $f'(x) = 2x$ and has a zero in $x_0 = 0$, i.e. $f'(0) = 0$.

The second derivative is $f''(x) = 2$, hence $f''(0) = 2 > 0$ and $f(0) = 0$ is a local minimum of f. It is attained in $x_0 = 0$.

This function has no maximum.

Extrema

The function

$$\begin{array}{rccl} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & g(x) := -x^2 \end{array}$$

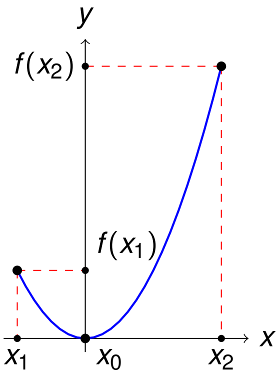
has a local maximum $g(0) = 0$ that is attained in $x_0 = 0$. This function has no minimum.

Extrema

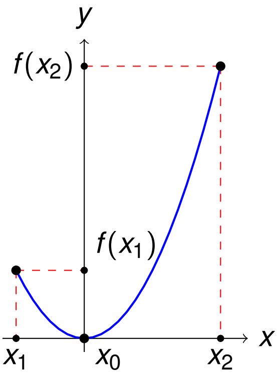
We now restrict the domain to closed intervals.

$$\begin{aligned} f: \quad [-1,2] &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) := x^2 . \end{aligned}$$

Since $0 \in [-1, 2]$, this function has a local minimum $f(0) = 0$ in $x_0 = 0$.



Extrema



We now have to evaluate f in the boundaries $x_1 = -1$ and $x_2 = 2$ of the interval $[-1, 2]$ and get

$$f(-1) = 1, \quad f(2) = 4$$

Hence

$$f(-1) > f(0)\,, \quad f(0) < f(2) \quad \text{and} \quad f(-1) < f(2)\,.$$

The function has a global minimum 0 in 0. It has a global maximum $f(2) = 4$ in $x_2 = 2$ and a local maximum $f(-1) = 1$ in $x_1 = -1$.

There is no exercise class this Friday, but there will be one tomorrow!

See you tomorrow!