Mathematics

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D-ARCH

September 26, 2023

Last time?

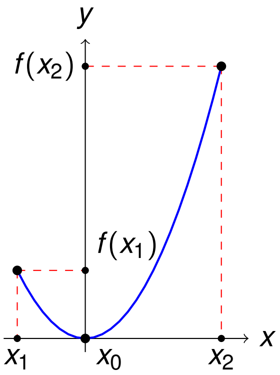
- Functions and extrema.

Extrema

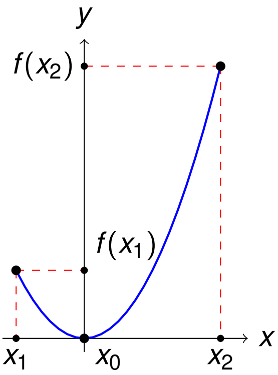
We now restrict the domain to closed intervals.

$$\begin{array}{rccl} f: & [-1,2] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) := x^2 \,. \end{array}$$

Since $0 \in [-1,2]$, this function has a local minimum $f(0) = 0$ in $x_0 = 0$.



Extrema



We now have to evaluate f in the boundaries $x_1 = -1$ and $x_2 = 2$ of the interval $[-1, 2]$ and get

$$f(-1) = 1, \quad f(2) = 4$$

Hence

$$f(-1) > f(0)\,, \quad f(0) < f(2) \quad \text{and} \quad f(-1) < f(2)\,.$$

The function has a global minimum 0 in 0. It has a global maximum $f(2) = 4$ in $x_2 = 2$ and a local maximum $f(-1) = 1$ in $x_1 = -1$.

Today?

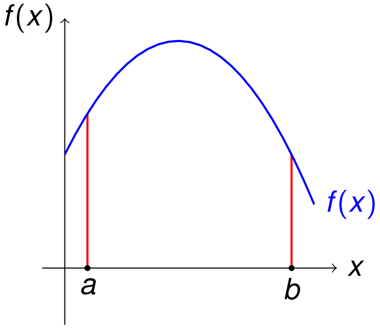
- Integrals

Integration

Given a function

$$f: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto f(x)$$

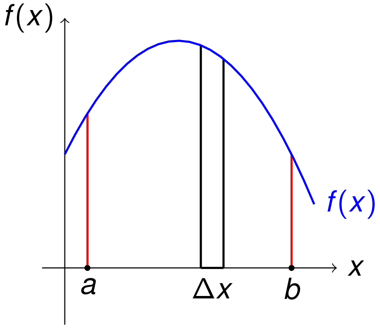
we compute the area between the graph $(x, f(x))$, $x \in [a, b]$, and the interval $[a, b]$, $a < b$.



Integral

We first cut the area in very thin stripes of width $\Delta x = \frac{b-a}{n}$ and approximate it with

$$\sum_{k=1}^{n} \Delta x \, f(\widetilde{x}_k)\,, \quad a + (k-1)\,\Delta x \leqslant \widetilde{x}_k \leqslant a + k\,\Delta x\,.$$

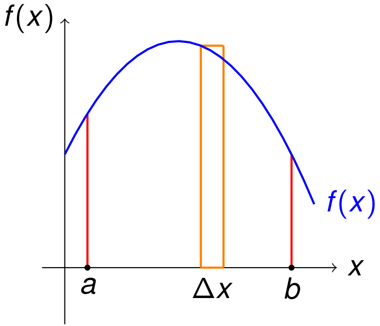


Integral

If

$$f(\widetilde{x}_k) = \max\{f(x) \mid a + (k-1)\,\Delta x \leqslant x \leqslant a + k\,\Delta x\}$$

we get an upper sum.

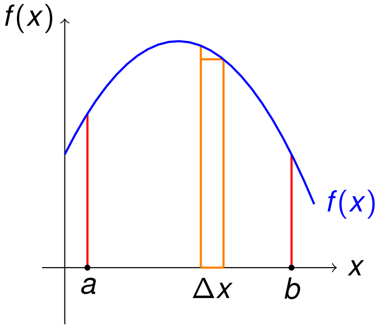


Integral

If

$$f(\widetilde{x}_k) = \min\{f(x) \mid a + (k-1)\,\Delta x \leqslant x \leqslant a + k\,\Delta x\}$$

we get a lower sum.



Integral

We define the integral of f on the interval $[a, b]$

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x\, f(\widetilde{x}_k)\,.$$

If the limit exists, then the limit of the upper sum equals the limit of the lower sum.

This integral is called the

Riemann integral.

Integral

Let $f : \mathrm{dom}(f) \to \mathbb{R}$. If $[a, b] \subseteq \mathrm{dom}(f)$, then the integral

$$\int_a^b f(x)\,dx$$

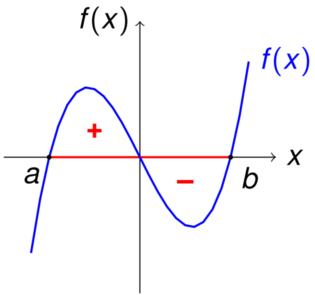
is the definite integral of f on $[a, b]$.

Integral

The integral

$$\int_a^b f(x)\, dx\,, \quad a < b\,,$$

determines the area between the x-axis and the graph of f on the interval $[a, b]$, where the area above the x-axis contributes to the integral with a positive sign and the area below the x-axis contributes with a negative sign.



Integral

Compare

$$\int_0^{\pi} \sin(x)\,dx$$

with

$$\int_0^{2\pi} \sin(x)\,dx$$

Integral

$$\int_0^{\pi} \sin(x)\, dx = 2$$

but

$$\int_0^{2\pi} \sin(x)\, dx = 0$$

Main theorem of integration theory

A function $F : \mathrm{dom}(F) \to \mathbb{R}$, $\mathrm{dom}(F) \subseteq \mathbb{R}$ that satisfies

$$\frac{d}{dx}F(x) = F'(x) = f(x)$$

is called the antiderivative of f. Two different antiderivatives of f differ by a constant. The set of all antiderivatives of f is called the indefinite integral of f.

$$\begin{aligned}\int f(x)\,dx &:= \left\{F(x) \mid \tfrac{d}{dx}F(x) = f(x)\right\} \\ &= \left\{F(x) + c \mid c \in \mathbb{R},\ F \text{ is an antiderivative of } f\right\}.\end{aligned}$$

Main theorem of integration theory

Given a function $f : \mathbb{R} \to \mathbb{R}$ with antiderivative

$$F : \mathbb{R} \to \mathbb{R}\,.$$

The definite integral of f on $[a, b]$ equals

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Integral

- $\int_0^2 3x^2 dx$
- $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx$
- $\int 2x \cos(x^2) dx$

Integral

- $$\int_0^2 3x^2 dx = x^3\Big|_0^2 = 8$$
- $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx = \sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = 2$$
- $$\int 2x \cos(x^2) dx = \sin(x^2) + C\,, \quad C \in \mathbb{R}$$

Have a nice week!