Mathematics

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Main theorem of integration theory

Given a function $f : \mathbb{R} \to \mathbb{R}$ with antiderivative

$$F : \mathbb{R} \to \mathbb{R} \, .$$

The definite integral of f on $[a, b]$ equals

$$\int_a^b f(x)\, dx = F(b) - F(a) \, .$$

Problem

How do we find the antiderivative F ?

How to integrate

Use the formulas for the derivatives:

product rule	$\rightsquigarrow$	integration by parts
chain rule	$\rightsquigarrow$	integration by substitution

How to integrate

Take the product rule

$$\big(f(x)\cdot g(x)\big)' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

and write it in a different way

$$f'(x)\cdot g(x) = \big(f(x)\cdot g(x)\big)' - f(x)\cdot g'(x)\,.$$

Integrate

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx\,.$$

How to integrate

The formula

$$\int f'(x) \cdot g(x)\,dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\,dx\,.$$

can be used to compute integrals of functions $f'(x) \cdot g(x)$.

The difficulty of this method consist in choosing the functions $f(x)$ and $g(x)$.

Example 1

Compute

$$\int x\, e^x \, dx$$

Example 1

Compute

$$\int x\, e^x\, dx\,.$$

Choose

$$f(x) = x\,, \quad g'(x) = e^x\,.$$

Hence

$$f'(x) = 1\,, \quad g(x) = e^x$$

and

$$\int x\, e^x\, dx = x\, e^x - \int 1 \cdot e^x\, dx$$
$$= x\, e^x - (e^x + C) \quad = (x-1)e^x + C\,.$$

Example 2

Compute

$$\int x^2 e^x \, dx$$

Example 2

Compute

$$\int x^2\, e^x\, dx$$

With

$$f(x) = x^2\,, \quad g'(x) = e^x$$
$$f'(x) = 2x\,, \quad g(x) = e^x\,,$$

$$\int x^2\, e^x\, dx = x^2\, e^x - 2\int x\, e^x\, dx$$
$$= x^2\, e^x - 2(x-1)e^x + C \quad = (x^2 - 2x + 2)e^x + C\,.$$

Here we use the result of the previous example or we apply the integration by parts on the integral $\int x\, e^x\, dx$.

Example 3

Compute

$$\int \cos(x) \cdot e^x \, dx$$

Example 3

Compute

$$\int \cos(x) \cdot e^x \, dx$$

The choice

$$\begin{aligned} f(x) &= \cos(x)\,, & g'(x) &= e^x\,, \\ f'(x) &= -\sin(x)\,, & g(x) &= e^x \end{aligned}$$

leads to

$$\begin{aligned} \int \cos(x) \cdot e^x \, dx &= \cos(x) \cdot e^x - \int (-\sin(x))\, e^x \, dx \\ &= \cos(x) \cdot e^x + \int \sin(x)\, e^x \, dx\,. \end{aligned}$$

Does it really help?

Example 3

Compute

$$\int \sin(x) \cdot e^x \, dx$$

The choice

$$h(x) = \sin(x)\,, \quad k'(x) = e^x\,,$$
$$h'(x) = \cos(x)\,, \quad k(x) = e^x\,,$$

yields

$$\int \sin(x)\, e^x \, dx = \sin(x) \cdot e^x - \int \cos(x)\, e^x \, dx$$

Example 3

$$\int \cos(x) \cdot e^x \, dx = \cos(x) \cdot e^x + \underbrace{\int \sin(x)\, e^x \, dx}$$

$$= \cos(x) \cdot e^x + \sin(x) \cdot e^x - \int \cos(x)\, e^x \, dx$$

Example 3

$$\int \cos(x) \cdot e^x \, dx = \cos(x) \cdot e^x + \underbrace{\int \sin(x)\, e^x \, dx}$$
$$= \cos(x) \cdot e^x + \sin(x) \cdot e^x - \int \cos(x)\, e^x \, dx$$

We are lucky!

$$\int \cos(x) \cdot e^x \, dx = \cos(x) \cdot e^x + \sin(x) \cdot e^x - \int \cos(x)\, e^x \, dx$$

Example 3

$$\int \cos(x) \cdot e^x \, dx = \cos(x) \cdot e^x + \sin(x) \cdot e^x - \int \cos(x) \, e^x$$

$$2 \int \cos(x) \cdot e^x \, dx = \cos(x) \cdot e^x + \sin(x) \cdot e^x$$

$$= \big(\cos(x) + \sin(x)\big) \cdot e^x$$

Result:

$$\int \cos(x) \cdot e^x \, dx = \frac{1}{2}\big(\cos(x) + \sin(x)\big) \cdot e^x$$

Example 4

Compute

$$\int \ln(x)\, dx$$

using integration by parts!

Yes, this is possible!

Example 4

Compute

$$\int \ln(x)\,dx$$

using integration by parts!

Hint:

$$\ln(x) = 1 \cdot \ln(x)$$

Example 4

Compute

$$\int \ln(x)\,dx = \int 1 \cdot \ln(x)\,dx$$

The choice

$$f'(x) = 1\,, \quad g(x) = \ln(x)\,,$$
$$f(x) = x\,, \quad g'(x) = \frac{1}{x}\,,$$

leads to

$$\begin{aligned}\int \ln(x)\,dx &= x\ln(x) - \int x \cdot \frac{1}{x}\,dx\\ &= x\ln(x) - \int 1\,dx\\ &= x\ln(x) - x + C\,, \quad C \in \mathbb{R}\,.\end{aligned}$$

See you tomorrow!