Mathematics

Cornelia Busch

D-ARCH

October 2, 2023

Problem

On a rainy day, the number of litre of water falling per hour on a square meter was given by the function $f : [0, 24] \to \mathbb{R}$

$$f(x) = \frac{1}{4}x - \frac{1}{96}x^2 ,$$

where $x \in \mathbb{R}$ gives the time in hours.

- When did it rain the heaviest?

Problem

On a rainy day, the number of litre of water falling per hour on a square meter was given by the function $f : [0, 24] \to \mathbb{R}$

$$f(x) = \frac{1}{4}x - \frac{1}{96}x^2\,,$$

where $x \in \mathbb{R}$ gives the time in hours.

- When did it rain the heaviest?

We determine the extrema.

$$f'(x) = \frac{1}{4} - \frac{1}{48}x = 0 \quad \Longleftrightarrow \quad x = 12$$

Since $f''(x) = -\frac{1}{48} < 0$, the function attains a local maximum in $x = 12$, with $f(12) = \frac{3}{4}$.
The function is defined on a closed interval. Hence we evaluate it on the boundaries.
$f(0) = f(24) = 0$.

It started raining at midnight and stopped 24 hours later. It rained heaviest at noon.

Problem

On a rainy day, the number of litre of water falling per hour on a square meter was given by the function $f : [0, 24] \to \mathbb{R}$

$$f(x) = \frac{1}{4}x - \frac{1}{96}x^2 ,$$

where $x \in \mathbb{R}$ gives the time in hours.

- How many water fell on a square meter on that day?

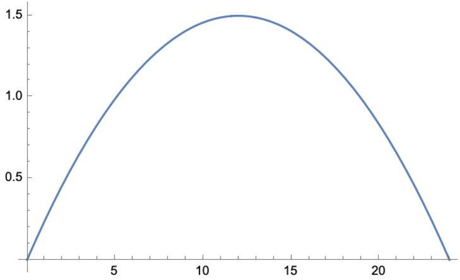
Problem

On a rainy day, the number of litre of water falling per hour on a square meter was given by the function $f : [0, 24] \to \mathbb{R}$

$$f(x) = \frac{1}{4}x - \frac{1}{96}x^2 ,$$

where $x \in \mathbb{R}$ gives the time in hours.

- How many water fell on a square meter on that day?



Problem

On a rainy day, the number of litre of water falling per hour on a square meter was given by the function $f : [0, 24] \to \mathbb{R}$

$$f(x) = \frac{1}{4}x - \frac{1}{96}x^2 ,$$

where $x \in \mathbb{R}$ gives the time in hours.

- How many water fell on a square meter on that day?

We integrate:

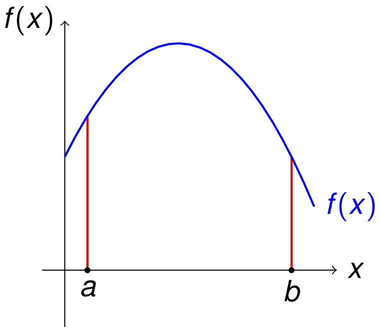
$$\int_0^{24} f(x)\,dx = \int_0^{24} \frac{1}{4}x - \frac{1}{96}x^2\,dx$$

Integration

Given a function

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

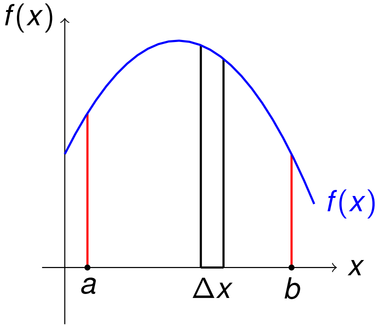
we compute the area between the graph $(x, f(x))$, $x \in [a, b]$, and the interval $[a, b]$, $a < b$.



Integral

We first cut the area in very thin stripes of width $\Delta x = \frac{b-a}{n}$ and approximate it with

$$\sum_{k=1}^{n} \Delta x \, f(\widetilde{x}_k)\,, \quad a + (k-1)\,\Delta x \leqslant \widetilde{x}_k \leqslant a + k\,\Delta x\,.$$

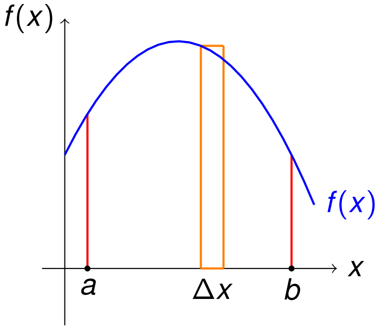


Integral

If

$$f(\widetilde{x}_k) = \max\{f(x) \mid a + (k-1)\,\Delta x \leqslant x \leqslant a + k\,\Delta x\}$$

we get an upper sum.

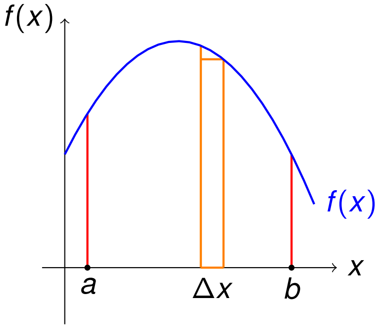


Integral

If

$$f(\widetilde{x}_k) = \min\{f(x) \mid a + (k-1)\,\Delta x \leqslant x \leqslant a + k\,\Delta x\}$$

we get a lower sum.



Integral

We define the integral of f on the interval $[a, b]$

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x\, f(\widetilde{x}_k)\,.$$

If the limit exists, then the limit of the upper sum equals the limit of the lower sum.

This integral is called the

Riemann integral.

Integral

Let $f : \mathrm{dom}(f) \to \mathbb{R}$. If $[a, b] \subseteq \mathrm{dom}(f)$, then the integral

$$\int_a^b f(x)\, dx$$

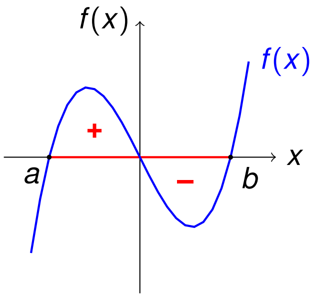
is the definite integral of f on $[a, b]$.

Integral

The integral

$$\int_a^b f(x)\,dx\,, \quad a < b\,,$$

determines the area between the x-axis and the graph of f on the interval $[a, b]$, where the area above the x-axis contributes to the integral with a positive sign and the area below the x-axis contributes with a negative sign.



Integral

Compare

$$\int_0^{\pi} \sin(x)\,dx$$

with

$$\int_0^{2\pi} \sin(x)\,dx$$

Integral

$$\int_0^{\pi} \sin(x)\, dx = 2$$

but

$$\int_0^{2\pi} \sin(x)\, dx = 0$$

Main theorem of integration theory

A function $F : \mathrm{dom}(F) \to \mathbb{R}$, $\mathrm{dom}(F) \subseteq \mathbb{R}$ that satisfies

$$\frac{d}{dx} F(x) = F'(x) = f(x)$$

is called the antiderivative of f. Two different antiderivatives of f differ by a constant. The set of all antiderivatives of f is called the indefinite integral of f.

$$\begin{aligned} \int f(x)\,dx &:= \left\{ F(x) \mid \tfrac{d}{dx} F(x) = f(x) \right\} \\ &= \left\{ F(x) + c \mid c \in \mathbb{R},\ F \text{ is an antiderivative of } f \right\}. \end{aligned}$$

Main theorem of integration theory

Given a function $f : \mathbb{R} \to \mathbb{R}$ with antiderivative

$$F : \mathbb{R} \to \mathbb{R}\,.$$

The definite integral of f on $[a, b]$ equals

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Integral

- $\int_0^2 3x^2 dx$
- $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx$
- $\int 2x \cos(x^2) dx$

Integral

- $\int_0^2 3x^2 dx = x^3\Big|_0^2 = 8$
- $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx = \sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = 2$
- $\int 2x \cos(x^2) dx = \sin(x^2) + C\,, \quad C \in \mathbb{R}$