Mathematics

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How to integrate

- Yesterday: Integration by parts
- Today: Integration by substitution

How to integrate

Use the formulas for the derivatives:

product rule $\rightsquigarrow$ integration by parts

chain rule $\rightsquigarrow$ integration by substitution

Integration by parts

$$\int f'(x) \cdot g(x)\, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\, dx\,.$$

or

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx\,.$$

Integration by substitution

Chain rule of differentiation
Let $I, J \subset \mathbb{R}$ be intervals. Let

$$f: \begin{array}{ccc} I & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array} \qquad \varphi: \begin{array}{ccc} J & \to & I \\ t & \mapsto & \varphi(t) \end{array}$$

We know that the derivative with respect to t of their composition is

$$\frac{d}{dt}\Big(f(\varphi(t))\Big) = f'(\varphi(t))\varphi'(t)\,.$$

Integration by substitution

Let $x := \varphi(t)$. Then

$$\int f\big(\varphi(t)\big)\,\varphi'(t)\,dt = \left(\int f(x)\,dx\right)_{x:=\varphi(t)}$$

and

$$\int_a^b f\big(\varphi(t)\big)\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx\,.$$

Hence we search for a substitution $x := \varphi(t)$ and determine the integral $\int f(x)\,dx$.

Difficulty: Find $x := \varphi(t)$.

Example 1

Compute

$$\int_a^b 2t \sin(t^2)\, dt$$

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$$\int_a^b 2t\sin(t^2)\,dt\,.$$

Substitute $x := \varphi(t) = t^2$ and get

$$\frac{dx}{dt} = 2t\,, \qquad \Rightarrow dx = 2t\,dt = \varphi'(t)\,dt$$

and

$$\int_a^b 2t\sin(t^2)\,dt = \int_{\varphi(a)}^{\varphi(b)} \sin(\varphi(t))\,\varphi'(t)\,dt = \int_{a^2}^{b^2} \sin(x)\,dx$$

$$= \big[-\cos(x)\big]_{a^2}^{b^2} = -\cos(b^2) - \big(-\cos(a^2)\big)$$

$$= \cos(a^2) - \cos(b^2)\,.$$

Example 2

Compute

$$J := \int (\cos t + \cos^3 t)\, dt$$

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Hint:

$$J = \int (1 + \cos^2 t)\, \cos t\, dt = \int (2 - \sin^2 t) \cos t\, dt\,.$$

Example 2

The substitution

$$\sin t := x\,, \quad \cos t\,dt = dx$$

yields

$$J = \int (2 - \sin^2 t) \cos t\,dt = \left(\int (2 - x^2)\,dx \right)_{x := \sin t}$$

$$= \left(2x - \frac{x^3}{3} \right)_{x := \sin t} + c$$

$$= 2 \sin t - \frac{1}{3} \sin^3 t + c\,.$$

Example 2

The definite integral

$$J_0 := \int_{\pi/6}^{\pi} (\cos t + \cos^3 t)\, dt$$

becomes with this substitution with $\sin \pi/6 = 1/2$ and $\sin \pi = 0$

$$J_0 = \int_{1/2}^{0} (2 - x^2)\, dx = \left(2x - \frac{x^3}{3}\right)\Big|_{1/2}^{0}$$

$$= -\left(1 - \frac{1}{24}\right) = -\frac{23}{24}\,.$$

Example 3

Compute

$$\int \frac{3x-1}{x^2-x+1}\,dx\,.$$

We know that $(x^2-x+1)' = 2x-1$.

With the numerator $2x-1$ we would substitute $v(x) = x^2-x+1$, $dv = (2x-1)dx$ and get

$$\begin{aligned}\int \frac{2x-1}{x^2-x+1}\,dx &= \int \frac{1}{v}\,dv = \ln|v| \\ &= \ln|x^2-x+1|\,.\end{aligned}$$

Example 3

We now transform the fraction in order to use this fact. Since

$$\frac{3x-1}{x^2-x+1} = \frac{3}{2}\cdot\frac{2x-\frac{2}{3}}{x^2-x+1} = \frac{3}{2}\cdot\frac{2x-1+\frac{1}{3}}{x^2-x+1}$$

$$= \frac{3}{2}\left(\frac{2x-1}{x^2-x+1} + \frac{\frac{1}{3}}{x^2-x+1}\right)$$

we get

$$\int \frac{3x-1}{x^2-x+1}\,dx = \frac{3}{2}\int\left(\frac{2x-1}{x^2-x+1} + \frac{\frac{1}{3}}{x^2-x+1}\right)dx$$

$$= \frac{3}{2}\ln|x^2-x+1| + \frac{1}{2}\int\frac{1}{x^2-x+1}\,dx$$

Example 3

For the integral

$$\int \frac{1}{x^2 - x + 1}\, dx$$

we search for a substitution that allows us to use

$$\int \frac{1}{u^2 + 1}\, du = \arctan u + C$$

Partial fraction decomposition

We consider functions

$$f(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$, $Q_m(x)$ are polynomials of degree $n < m$. Let $\alpha_1, \ldots, \alpha_m$ be the zeros of $Q_m(x)$, i.e.,

$$Q_m(x) = (x - \alpha_1) \cdot \ldots \cdot (x - \alpha_m)$$

and let $P_n(\alpha_i) \neq 0$ for $i = 1, \ldots, m$, i.e., a zero α_i of $Q_m(x)$ is not a zero of $P_n(x)$.

Partial fraction decomposition

If $\alpha_1, \ldots, \alpha_m$ are distinct constants, then we make the ansatz

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_m}{x - \alpha_m}.$$

If the multiplicity of α_2 is $\ell > 1$, then we make the ansatz

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x - \alpha_1} + \frac{B_1}{x - \alpha_2} + \frac{B_2}{(x - \alpha_2)^2} + \cdots + \frac{B_\ell}{(x - \alpha_2)^\ell} + \frac{C_1}{x - \alpha_3} + \ldots$$

Partial fraction decomposition

We determine

$$\int f(x)\,dx = \int \frac{P_n(x)}{Q_m(x)}\,dx$$
$$= \int \frac{A_1}{x-\alpha_1}\,dx + \int \frac{B_1}{x-\alpha_2}\,dx + \cdots + \int \frac{B_\ell}{(x-\alpha_2)^\ell}\,dx + \ldots$$

using

$$\int \frac{A}{x-\alpha}\,dx = A \cdot \ln|x-\alpha| + c$$

and for $n \neq 1$

$$\int \frac{A}{(x-\alpha)^n}\,dx = \frac{-A}{(n-1)}(x-\alpha)^{-(n-1)} + c$$

Example 1

In order to compute

$$\int \frac{-1}{x^2 + 5x + 6}\, dx$$

we first solve

$$x^2 + 5x + 6 = 0 \quad \Leftrightarrow \quad x_{1,2} = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm 1}{2}$$

hence

$$x_1 = -2\,, \qquad x_2 = -3$$

and

$$x^2 + 5x + 6 = (x + 2)(x + 3)\,.$$

Example 1

To determine constants $A, B \in \mathbb{R}$ such that

$$\frac{-1}{x^2+5x+6} = \frac{-1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

we compute

$$\begin{aligned}
\frac{A}{x+2} + \frac{B}{x+3} &= \frac{A(x+3)}{(x+2)(x+3)} + \frac{B(x+2)}{(x+2)(x+3)} \\
&= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)} \\
&= \frac{(A+B)x+3A+2B}{(x+2)(x+3)} \\
&\overset{!}{=} \frac{-1}{(x+2)(x+3)}
\end{aligned}$$

Example 1

Looking at

$$\frac{-1}{(x+2)(x+3)} = \frac{(A+B)x+3A+2B}{(x+2)(x+3)}$$

we get by comparison of the coefficients a system of equations for A and B:

$$\begin{aligned} A+B &= 0 \\ 3A+2B &= -1 \end{aligned}$$

that has the solutions

$$A=-1\,, \qquad B=1\,.$$

Hence

$$\frac{-1}{x^2+5x+6} = \frac{-1}{(x+2)(x+3)} = \frac{-1}{x+2} + \frac{1}{x+3}$$

Example 1

$$\int \frac{-1}{x^2+5x+6}\,dx = \int \frac{-1}{x+2}\,dx + \int \frac{1}{x+3}\,dx$$
$$= -\ln|x+2| + \ln|x+3| + C\,, \qquad C \in \mathbb{R}$$
$$= \ln\left|\frac{x+3}{x+2}\right| + C\,.$$

Example 2

Compute

$$\int \frac{x}{(x-1)^2\,(x+1)}\,dx\,.$$

We determine A_1, A_2 and B such that

$$\frac{x}{(x-1)^2\,(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B}{x+1}$$

Example 2

$$
\begin{aligned}
\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{B}{x+1} &= \frac{A_1(x-1)(x+1)+A_2(x+1)+B(x-1)^2}{(x-1)^2\,(x+1)} \\
&= \frac{A_1(x^2-1)+A_2(x+1)+B(x^2-2x+1)}{(x-1)^2\,(x+1)} \\
&= \frac{(A_1+B)x^2+(A_2-2B)x-A_1+A_2+B}{(x-1)^2\,(x+1)} \\
&\overset{!}{=} \frac{x}{(x-1)^2\,(x+1)}
\end{aligned}
$$

Example 2

The equation

$$\frac{x}{(x-1)^2\,(x+1)} \stackrel{!}{=} \frac{(A_1+B)x^2+(A_2-2B)x-A_1+A_2+B}{(x-1)^2\,(x+1)}$$

yields the system

$$\begin{aligned} A_1+B &= 0 \\ A_2-2B &= 1 \\ -A_1+A_2+B &= 0 \end{aligned}$$

With $B=-A_1$ we get

$$\begin{aligned} A_2+2A_1 &= 1 \\ A_2-2A_1 &= 0 \end{aligned}$$

hence $A_2 = 2A_1$ and $2A_2 = 1$ yields the solution

$$A_1 = \frac{1}{4}\,, \quad A_2 = \frac{1}{2}\,, \quad B = \frac{-1}{4}\,.$$

Example 2

$$\int \frac{x}{(x-1)^2(x+1)}\,dx = \int \frac{\left(\frac{1}{4}\right)}{x-1} + \frac{\left(\frac{1}{2}\right)}{(x-1)^2} + \frac{-\left(\frac{1}{4}\right)}{x+1}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{(x-1)^2}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{1}{2}\cdot\frac{-1}{(x-1)} - \frac{1}{4}\ln|x+1| + c$$

$$= \frac{-1}{2(x-1)} + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + c$$

The area function

We define the area function

$$F_a : [a, \infty) \to \mathbb{R}$$

of a non-negative function $f : \mathbb{R} \to \mathbb{R}_0^+$ to be

$$F_a(x) = \int_a^x f(t)dt\,,$$

the area between the graph of f and the x-axis on the interval $[a, x]$. This function satisfies $F_a(a) = 0$ and for $b > a$

$$F_b(x) = F_a(x) - F_a(b)$$

It can be shown that

$$F_a'(x) = f(x)\,.$$

A problem

Find three different methods to compute the following integral.

$$\int \sin(x)\cos(x)\,dx$$

You will see one of the three solutions in the exercises.

This week there will be an exercise class on Friday!

Have a nice week!