Mathematics
Complex numbers

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Last week

How to integrate?

- Integration by parts
- Integration by substitution
- Partial fraction decomposition

Problem

Problem
Find three different methods to compute the following integral.

$$\int \sin(x)\cos(x)\,dx$$

Solution 1: Trigonometry

We use

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

$$\begin{aligned}\int \sin(x)\cos(x)\,dx &= \int \frac{1}{2}\sin(2x)\,dx \\ &= \frac{1}{2}\cdot\frac{(-1)}{2}\cos(2x) + C \\ &= \frac{-\cos(2x)}{4} + C\,, \quad C \in \mathbb{R}\end{aligned}$$

Solution 2: Integration by parts

The choice

$$f'(x) = \sin(x) \qquad g(x) = \cos(x)$$
$$f(x) = -\cos(x) \qquad g'(x) = -\sin(x)$$

yields

$$\int \sin(x)\cos(x)\,dx = -\cos^2(x) - \int \cos(x)\sin(x)\,dx$$

Hence

$$\begin{aligned}\int \sin(x)\cos(x)\,dx &= -\frac{1}{2}\cos^2(x) + c \\ &= -\frac{1}{2}\left(\frac{1+\cos(2x)}{2}\right) + c = -\frac{1}{4} - \frac{\cos(2x)}{4} + c \\ &= -\frac{\cos(2x)}{4} + C\,, \quad C \in \mathbb{R}\end{aligned}$$

Solution 2: Substitution

We substitute

$$y = \sin(x)\,, \quad dy = \cos(x)dx$$

and get

$$\begin{aligned}
\int \sin(x)\cos(x)\,dx &= \int y\,dy = \frac{1}{2}y^2 + c \\
&= \frac{1}{2}\sin^2(x) + c = \frac{1}{2}(1 - \cos^2(x)) + c \\
&= \frac{1}{2}\left(1 - \frac{1}{2}(1 + \cos(2x))\right) + c = \frac{1}{4} - \frac{\cos(2x)}{4} + c \\
&= -\frac{\cos(2x)}{4} + C\,, \quad C \in \mathbb{R}
\end{aligned}$$

Today?

- Complex numbers

Introduction

We consider the equation

$$ax^2 + bx + c = 0\,,$$

with $a, b, c \in \mathbb{R}$. If $a \neq 0$, the solutions of this quadratic equation are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\,.$$

We get

$$x_1 = x_2 \;\Leftrightarrow\; b^2 - 4ac = 0$$

and

$$x_1 = -x_2 \;\Leftrightarrow\; b = 0\,.$$

Example

The solutions of the equation

$$x^2 - 5x + 6 = 0$$

are

$$x_1 = \frac{5 + \sqrt{25 - 24}}{2} = 3 \quad \text{and} \quad x_2 = \frac{5 - \sqrt{25 - 24}}{2} = 2\,.$$

Hence

$$x^2 - 5x + 6 = (x - 3)(x - 2)\,.$$

Another example

The solutions of the equation

$$x^2 - 1 = 0$$

are the solutions of $x^2 = 1$:

$$x_1 = 1 \quad \text{and} \quad x_2 = -1\,.$$

Example

The solution of

$$x^2 + 1 = 0$$

are the solutions of $x^2 = -1$:

$$x_1 = \sqrt{-1} \quad \text{and} \quad x_2 = -\sqrt{-1}\,.$$

Problem: There is no $x \in \mathbb{R}$ with $x^2 = -1$.

Definition of i

We define i such that $i^2 = -1$.

Then the solutions of

$$x^2 + 1 = 0$$

are

$$x_1 = i \quad \text{and} \quad x_2 = -i\,.$$

The number i that solves $x^2 + 1 = 0$ is called the imaginary unit.

Complex solutions

The equation

$$ax^2 + bx + c = 0\,,$$

with $a, b, c \in \mathbb{R}$, $a \neq 0$, has no real solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if and only if

$$b^2 - 4ac < 0\,.$$

In this case the solutions are

$$x_1 = \frac{-b}{2a} + i\,\frac{\sqrt{-b^2 + 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b}{2a} - i\,\frac{\sqrt{-b^2 + 4ac}}{2a}\,.$$

Definition of $\mathbb{C}$

We define the field $\mathbb{C}$ of complex numbers to be the smallest field[1] that is an extension of the field of real numbers $\mathbb{R}$ and contains the solution i of the equation $x^2 + 1 = 0$. The elements of the field $\mathbb{C}$ are called complex numbers.

Every complex number $z \in \mathbb{C}$ can be written as

$$z = x + iy \quad \text{with } x, y \in \mathbb{R}\,.$$

The pair $(x, y) \in \mathbb{R} \times \mathbb{R}$ is unique.

[1] Examples of fields are the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$.

Complex conjugation

Let

$$z = x + iy \in \mathbb{C}\,,$$

$x, y \in \mathbb{R}$, be a complex number. We define the complex conjugate $\overline{z}$ of z to be

$$\overline{z} = x - iy\,.$$

We define the complex conjugation to be the mapping

$$\overline{\cdot}\,:\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \overline{z} \end{array}$$

that maps z to its complex conjugate.

Properties of the complex conjugation

The complex conjugation has the following properties.

1. For all $z \in \mathbb{C}$,
$$\overline{\overline{z}} = z \,.$$
2. For any $x \in \mathbb{R}$,
$$\overline{x} = x \,.$$
Hence the complex conjugation acts as the identity on $\mathbb{R}$.
3. For any $y \in \mathbb{R}$,
$$\overline{iy} = -iy \,.$$

Addition of complex numbers

We define an *addition* $+$ in $\mathbb{C}$:

$$\begin{array}{rccc} + : & \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (z_1, z_2) & \longmapsto & z_1 + z_2 \end{array}$$

where for $z_1 = x_1 + iy_1 \in \mathbb{C}$, $z_2 = x_2 + iy_2 \in \mathbb{C}$, with $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Product of complex numbers

We define a product

$$\begin{array}{rccc} \cdot : & \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (z_1, z_2) & \longmapsto & z_1 \cdot z_2 \end{array}$$

where for $z_1 = x_1 + iy_1 \in \mathbb{C}$, $z_2 = x_2 + iy_2 \in \mathbb{C}$, with $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1 x_2 + x_1 iy_2 + iy_1 x_2 + iy_1 iy_2 \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2) \end{aligned}$$

Real and imaginary part

Let $z \in \mathbb{C}$ be a complex number and $\overline{z} \in \mathbb{C}$ its complex conjugate. We call

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$$

the real part of z and

$$\mathrm{Im}(z) = -\frac{i}{2}(z - \overline{z})$$

the imaginary part of z.
For any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, the number $i\alpha$ is called imaginary.

Problem

Show that $z = x + iy$ and $\overline{z} = x - iy$ are the solutions of

$$z^2 - 2x\,z + x^2 + y^2 = 0\,.$$

Definitions

Let $z \in \mathbb{C}$ be a complex number and $\overline{z} \in \mathbb{C}$ its complex conjugate. We define the absolute value or modulus of $z \in \mathbb{C}$ to be

$$|z| = \sqrt{z\,\overline{z}}\,.$$

If $z = x + iy$ for some $x, y \in \mathbb{R}$, then

$$|z| = \sqrt{x^2 + y^2}\,.$$

If $z = x \in \mathbb{R}$, then

$$|z| = \sqrt{x^2} = |x|\,,$$

i.e. the absolute value of a real number equals its absolute value as a complex number.

Complex plane

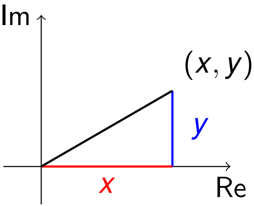
We know that for each complex $z \in \mathbb{C}$ there are $x, y \in \mathbb{R}$ such that $z = x + iy$. The mapping

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{C} \\ (x, y) & \longmapsto & z = x + iy \end{array}$$

is invertible

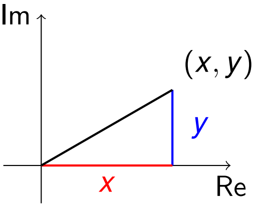
$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ z & \longmapsto & (\mathrm{Re}(z), \mathrm{Im}(z)) \, . \end{array}$$

It describes an isomorphism (a one-to-one correspondence) between the complex field $\mathbb{C}$ and the real plane $\mathbb{R}^2$.



$$(x, y) \in \mathbb{R}^2 \longleftrightarrow z = x + iy \in \mathbb{C}$$

Complex plane



$$(x, y) \in \mathbb{R}^2 \longleftrightarrow z = x + iy \in \mathbb{C}$$

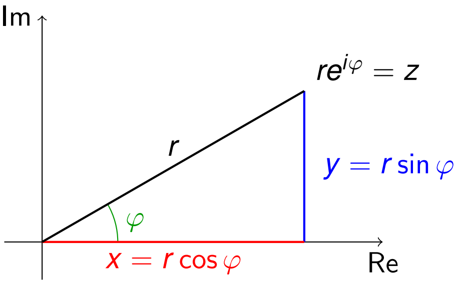
We call the x-axis the real axis Re and the y-axis the imaginary axis Im.
The plane is called complex plane.
It is sometimes known as Argand plane or Gauß plane.

Polar form

In analogy to the polar coordinates in the plane, there is a unique polar representation for every complex number $z \in \mathbb{C}$, $z \neq 0$. It is

$$z = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$$

with $r \in \mathbb{R}_+$ and $\varphi \in]-\pi, \pi]$.



This week there will be no exercise class on Friday!

See you tomorrow!