Mathematics
Complex numbers

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D-ARCH

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History of complex numbers

Raffael Bombelli and Geralomo Cardano (XVI century) use complex numbers to solve cubic equations.

René Descartes (1637) introduces the concept of imaginary number.

Caspar Wessel (1797) and Rowan William Hamilton (1833) make a formally correct definition of complex numbers.

Leonhard Euler (1748, Introductio in analysin infinitorum) relation between the exponential function and the trigonometric functions: the Euler formula.

Carl Friedrich Gauß (1777-1855) Fundamental theorem of algebra

Bernhard Riemann, Augustin-Louis Cauchy, Karl Weierstraß (XIX century) complex analysis.

Yesterday

Definition of complex numbers.

- Complex plane: cartesian coordinates.

Today?

- Complex numbers: polar form

Complex plane

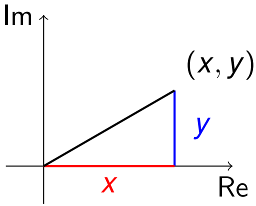
We know that for each complex $z \in \mathbb{C}$ there are $x, y \in \mathbb{R}$ such that $z = x + iy$. The mapping

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{C} \\ (x, y) & \longmapsto & z = x + iy \end{array}$$

is invertible

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ z & \longmapsto & (\mathrm{Re}(z), \mathrm{Im}(z)) \, . \end{array}$$

It describes an isomorphism (a one-to-one correspondence) between the complex field $\mathbb{C}$ and the real plane $\mathbb{R}^2$.



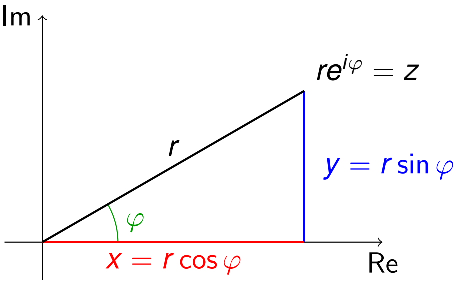
$$(x, y) \in \mathbb{R}^2 \longleftrightarrow z = x + iy \in \mathbb{C}$$

Polar form

In analogy to the polar coordinates in the plane, there is a unique polar representation for every complex number $z \in \mathbb{C}$, $z \neq 0$. It is

$$z = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$$

with $r \in \mathbb{R}_+$ and $\varphi \in]-\pi, \pi]$.



Polar form

Given $z = re^{i\varphi}$, we get

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi\,.$$

For a given $z = x + iy \in \mathbb{C}$ we see that the modulus (absolute value) of z

$$|z| = r = |(x, y)| = \sqrt{x^2 + y^2}$$

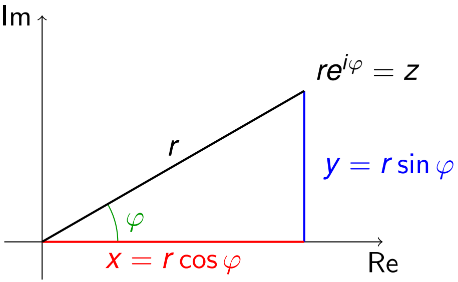
is the distance between (x, y) and the origin.

Polar form

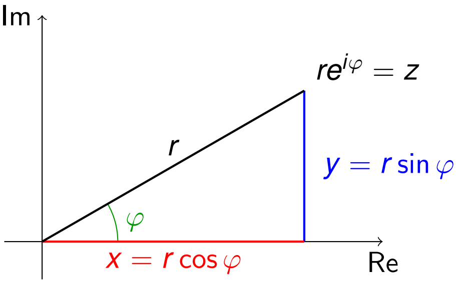
If $z = x + iy \in \mathbb{C}$, $z \neq 0$, satisfies $x > 0$, then the argument is

$$\varphi = \arctan\left(\frac{y}{x}\right),$$

where $\arctan : \mathbb{R} \to]-\frac{\pi}{2}, \frac{\pi}{2}[$ is the inverse of the tangent function.

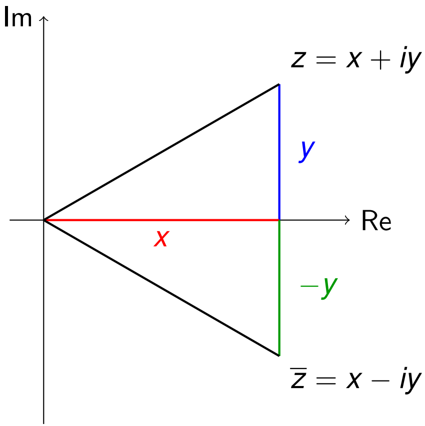


Polar form



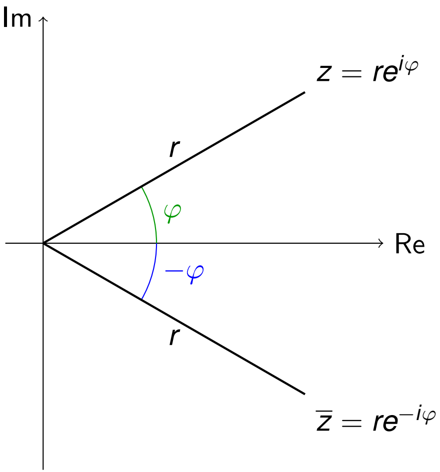
$$\varphi = \begin{cases} -\pi + \arctan(\frac{y}{x}) & \text{if } x < 0, y < 0, \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0, \\ \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \frac{\pi}{2} & \text{if } x = 0, y > 0, \\ \pi + \arctan(\frac{y}{x}) & \text{if } x < 0, y \geqslant 0\,. \end{cases}$$

Complex conjugation in the plane



The complex conjugation corresponds to a reflection on the real axis.

Complex conjugation in the plane

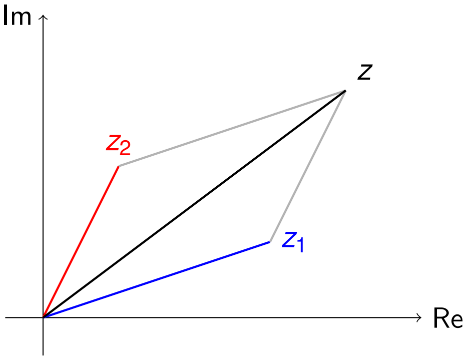


Hence

$$\overline{re^{i\varphi}} = re^{-i\varphi} .$$

Addition of complex numbers

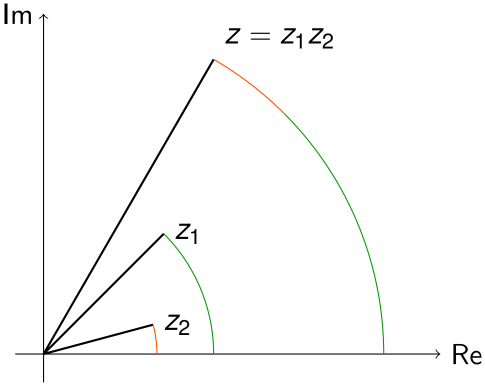
The addition of complex numbers corresponds to an addition of vectors. We choose the cartesian form. Let $z_1, z_2 \in \mathbb{C}$, then $z = z_1 + z_2$ is determined as follows.



Product of complex numbers

We choose the polar form. The product of $z_1 = r_1 e^{i\varphi_1}$ and $z_2 = r_2 e^{i\varphi_2}$ is

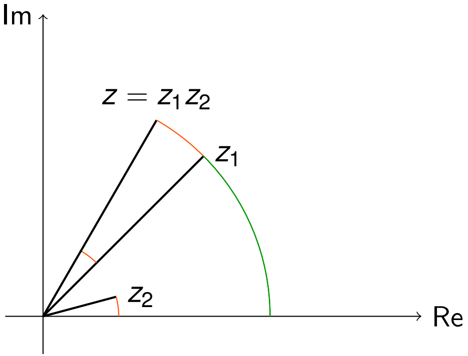
$$z = z_1 \cdot z_2 = r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2\, e^{i\varphi_1} e^{i\varphi_2} = r_1 r_2\, e^{i(\varphi_1+\varphi_2)} = re^{i\varphi}$$



Rotation in the complex plane

The multiplication with a number of modulus $r = 1$ represents a rotation in the complex plane. Indeed the product of $z_1 = r_1 e^{i\varphi_1}$ and $z_2 = e^{i\varphi_2}$ is

$$z = z_1 \cdot z_2 = r_1 e^{i\varphi_1} \cdot e^{i\varphi_2} = r_1\, e^{i(\varphi_1+\varphi_2)}$$



Exponent

Compute for some $n \in \mathbb{Z}$ the n-th power z^n of $z = re^{i\varphi}$.

$$z^n = r^n \left(e^{i\varphi}\right)^n = r^n \, e^{i\,n\varphi} \, .$$

If we set $r = 1$, then we have $z = e^{i\varphi}$ and

$$z^n = \left(e^{i\varphi}\right)^n = e^{i\,n\varphi} \, .$$

Depending on n and on $\varphi \in]-\pi, \pi]$, we might have $n\varphi \notin]-\pi, \pi]$. For example if $\varphi = \frac{\pi}{3}$, then $4\varphi = \frac{4\pi}{3} \notin]-\pi, \pi]$, but

$$e^{i\frac{4\pi}{3}} = e^{i\left(\frac{-2\pi}{3} + 2\pi\right)}$$

since

$$e^{i\varphi} = e^{i(\varphi + 2k\pi)} \quad \text{for any } \varphi \in \mathbb{R} \text{ and any } k \in \mathbb{Z} \, .$$

Root

An n-th root of $re^{i\varphi}$ is a solution of the equation

$$z^n = re^{i\varphi} = re^{i\,(\varphi + k\,2\pi)}\,, \quad k \in \mathbb{Z}\,.$$

Hence every

$$z_k = r^{\frac{1}{n}} e^{i\,(\frac{\varphi}{n} + k\,\frac{2\pi}{n})}\,, \quad k \in \mathbb{Z}\,,$$

is a solution of $z^n = 1 = e^{i\,k\,2\pi}$. Since

$$z_k = r^{\frac{1}{n}} e^{i\,(\frac{\varphi}{n} + k\,\frac{2\pi}{n})} = r^{\frac{1}{n}} e^{i\,(\frac{\varphi}{n} + (k+n)\,\frac{2\pi}{n})} = z_{k+n}$$

only $z_0, z_1, \ldots, z_{n-1}$ are pairwise different.

Roots of 1

An n-th root of 1 is a solution of the equation

$$z^n = 1 = e^{i\,k\,2\pi}\,, \quad k \in \mathbb{Z}\,.$$

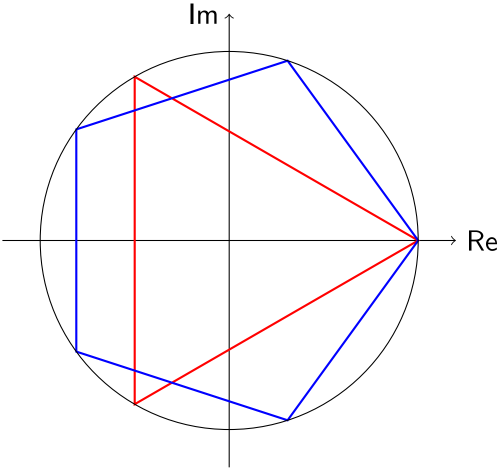
Hence every

$$z_k = e^{i\,k\,\frac{2\pi}{n}}\,, \quad k \in \mathbb{Z}\,,$$

is a solution of $z^n = 1 = e^{i\,k\,2\pi}$.

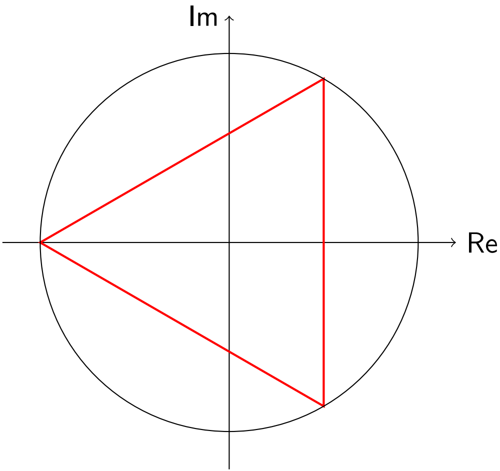
Roots of 1

In the complex plane the solutions of the *n*-th root of 1 lie on a regular polygon with *n* vertices.



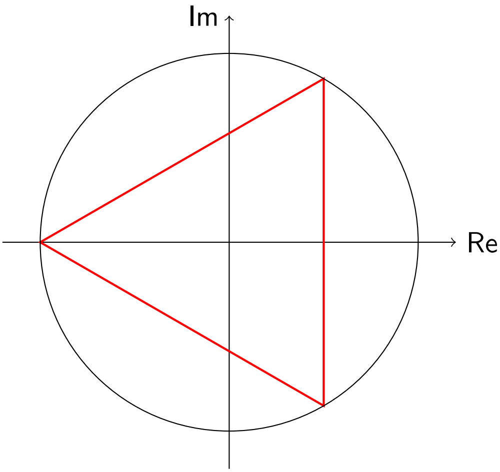
Question

What are the vertices of the following regular polygon?



Question

The vertices of the following regular polygon are the third roots of -1.



Proof by induction

Prove de Moivre's Theorem

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

by induction on n.

- You first prove the case $n = 1$ and
- then you show that if the claim holds for $n - 1$, then it also holds for n.

This week there will be no exercise class on Friday!

Have a nice week!