Mathematics
Complex numbers

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Last week

Definition of complex numbers.

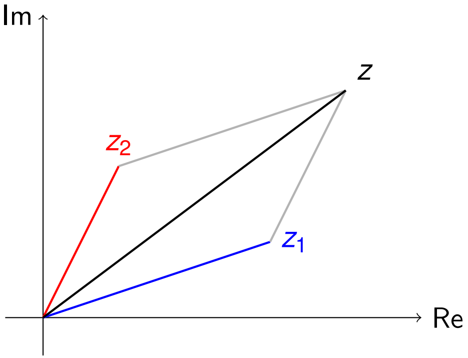
- Complex plane: cartesian coordinates.
- Complex numbers: polar form

Today?

- Powers and roots.

Addition of complex numbers

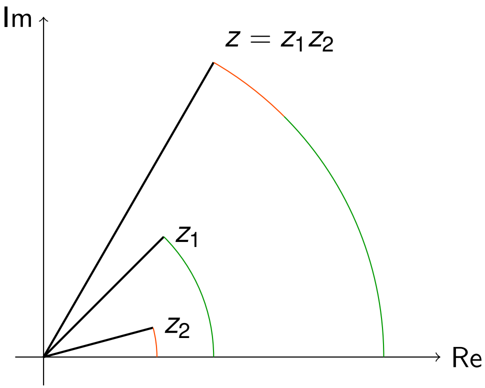
The addition of complex numbers corresponds to an addition of vectors. We choose the cartesian form. Let $z_1, z_2 \in \mathbb{C}$, then $z = z_1 + z_2$ is determined as follows.



Product of complex numbers

We choose the polar form. The product of $z_1 = r_1 e^{i\varphi_1}$ and $z_2 = r_2 e^{i\varphi_2}$ is

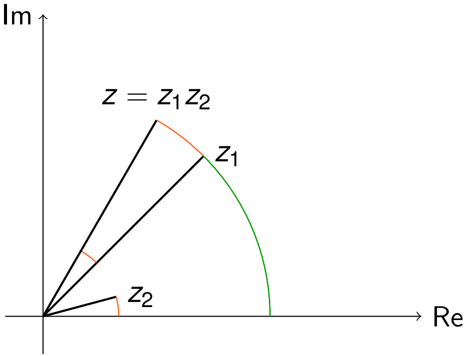
$$z = z_1 \cdot z_2 = r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2 \, e^{i\varphi_1} e^{i\varphi_2} = r_1 r_2 \, e^{i(\varphi_1 + \varphi_2)} = r e^{i\varphi}$$



Rotation in the complex plane

The multiplication with a number of modulus $r = 1$ represents a rotation in the complex plane. Indeed the product of $z_1 = r_1 e^{i\varphi_1}$ and $z_2 = e^{i\varphi_2}$ is

$$z = z_1 \cdot z_2 = r_1 e^{i\varphi_1} \cdot e^{i\varphi_2} = r_1\, e^{i(\varphi_1 + \varphi_2)}$$



Exponent

Compute for some $n \in \mathbb{Z}$ the n-th power z^n of $z = re^{i\varphi}$.

$$z^n = r^n \left(e^{i\varphi}\right)^n = r^n\, e^{i\,n\varphi}\,.$$

If we set $r = 1$, then we have $z = e^{i\varphi}$ and

$$z^n = \left(e^{i\varphi}\right)^n = e^{i\,n\varphi}\,.$$

Depending on n and on $\varphi \in]-\pi, \pi]$, we might have $n\varphi \notin]-\pi, \pi]$. For example if $\varphi = \frac{\pi}{3}$, then $4\varphi = \frac{4\pi}{3} \notin]-\pi, \pi]$, but

$$e^{i\frac{4\pi}{3}} = e^{i(\frac{-2\pi}{3}+2\pi)}$$

since

$$e^{i\varphi} = e^{i(\varphi+2k\pi)} \quad \text{for any } \varphi \in \mathbb{R} \text{ and any } k \in \mathbb{Z}\,.$$

Root

An n-th root of $re^{i\varphi}$ is a solution of the equation

$$z^n = re^{i\varphi} = re^{i\,(\varphi + k\,2\pi)}\,, \quad k \in \mathbb{Z}\,.$$

Hence every

$$z_k = r^{\frac{1}{n}} e^{i\,(\frac{\varphi}{n} + k\,\frac{2\pi}{n})}\,, \quad k \in \mathbb{Z}\,,$$

is a solution of $z^n = 1 = e^{i\,k\,2\pi}$. Since

$$z_k = r^{\frac{1}{n}} e^{i\,(\frac{\varphi}{n} + k\,\frac{2\pi}{n})} = r^{\frac{1}{n}} e^{i\,(\frac{\varphi}{n} + (k+n)\,\frac{2\pi}{n})} = z_{k+n}$$

only $z_0, z_1, \dots, z_{n-1}$ are pairwise different.

Roots of 1

An n-th root of 1 is a solution of the equation

$$z^n = 1 = e^{i\,k\,2\pi}\,, \quad k \in \mathbb{Z}\,.$$

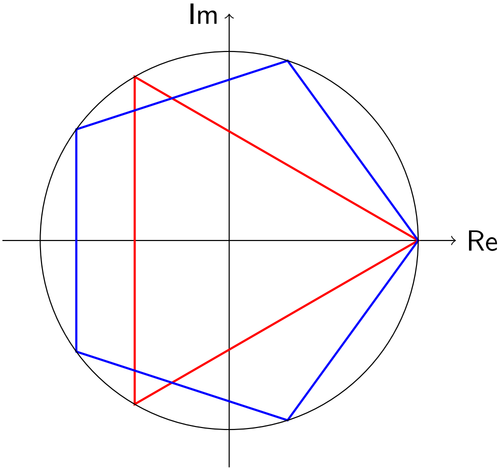
Hence every

$$z_k = e^{i\,k\,\frac{2\pi}{n}}\,, \quad k \in \mathbb{Z}\,,$$

is a solution of $z^n = 1 = e^{i\,k\,2\pi}$.

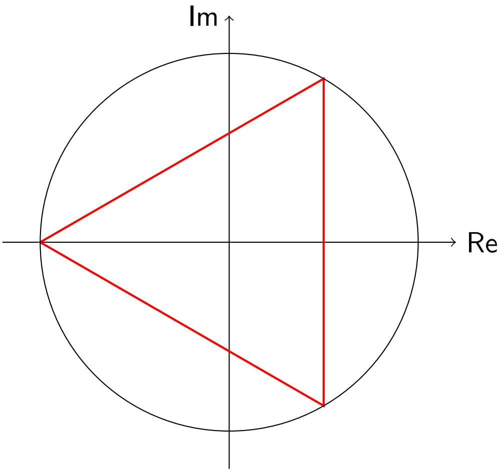
Roots of 1

In the complex plane the solutions of the *n*-th root of 1 lie on a regular polygon with *n* vertices.



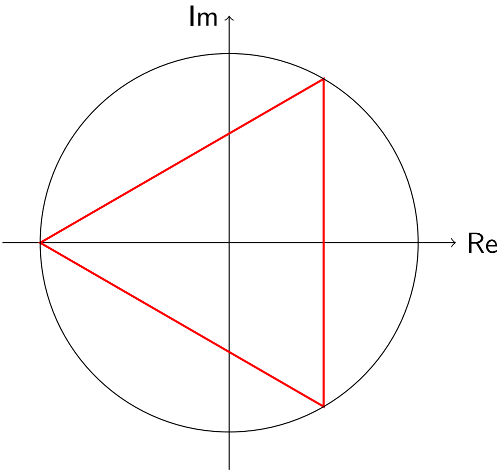
Question

What are the vertices of the following regular polygon?



Question

The vertices of the following regular polygon are the third roots of -1.



Proof by induction

Prove de Moivre's Theorem

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

by induction on n.

- You first prove the case $n = 1$ and
- then you show that if the claim holds for $n - 1$, then it also holds for n.

Next chapter!