Mathematics
Multivariable functions

Cornelia Busch

D-ARCH

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Last time

Complex numbers

Today

- Multivariable functions: Introduction
- Scalar fields

Introduction

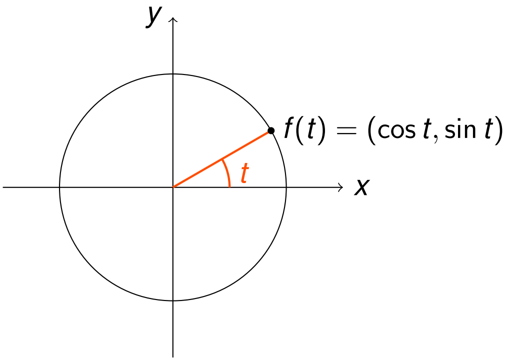
Curves	$f : \mathbb{R} \to \mathbb{R}^n$	Length of curves, line integrals, curvature
Surfaces	$f : \mathbb{R}^2 \to \mathbb{R}^n$	Areas of surfaces, surface integrals, flux through surfaces, curvature
Scalar fields	$f : \mathbb{R}^n \to \mathbb{R}$	Maxima and minima, Lagrange multipliers, directional derivatives
Vector fields	$f : \mathbb{R}^m \to \mathbb{R}^n$	Any of the operations of vector calculus, gradient, divergence, curl

Curve in the plane

The curve given by

$$\begin{array}{rccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & t & \longmapsto & f(t) := (\cos t, \sin t) \end{array}$$

is the unit circle in the plane.

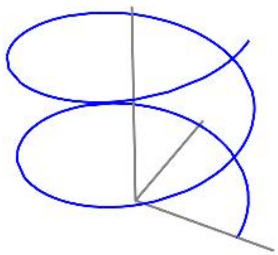


Curve in the 3-dimensional space

The curve parametrized with

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & t & \longmapsto & (\cos t, \sin t, t) \end{array}$$

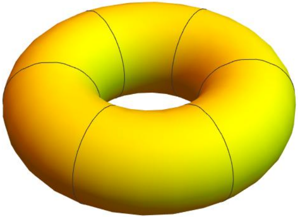
is a line that "screws upwards".



2-dim surface in a 3-dimensional space

Parametrisation of a torus: $f : [0, 2\pi[\times[0, 2\pi[\to \mathbb{R}^3$

$$(\theta, \varphi) \longmapsto \Big(\big(R + r\cos(\theta)\big)\cos(\varphi),\ \big(R + r\cos(\theta)\big)\sin(\varphi),\ \sin(\theta) \Big)$$



Scalar fields

In this section we consider functions

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

that map points $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ to scalars $f(x_1, \ldots, x_n)$. If $D \subset \mathbb{R}^n$, then the graph

$$\{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in D\}$$

describes a surface over D.

The function f may represent the metres above sea level of a point on a map or the temperature at a point in a space.

Level set

The level set of the function

$$f: \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

to the level $c \in \mathbb{R}$ is the set

$$f^{-1}(c) := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) = c\right\}.$$

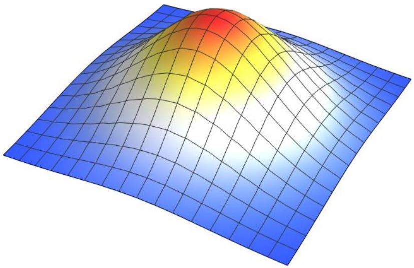
On the examples above it corresponds to the points at the same altitude or with the same temperature.

Example

Consider the function

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \end{array}$$

What are its level sets?



Example

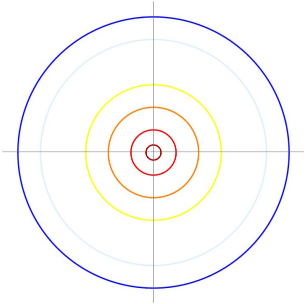
The level set of f to the level $c \in \mathbb{R}$ is the set of points $(x, y) \in \mathbb{R}^2$ that satisfy $f(x, y) = c$, where c is a constant. Since

$$c = e^{-(x^2+y^2)} \quad \Leftrightarrow \quad x^2 + y^2 = C\,,$$

where $C \in \mathbb{R}$ is a constant, we get the level lines

$$x^2 + y^2 = C = r^2\,.$$

These are circles with radius r centred in $(0, 0)$.



Partial derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in n variables. We fix a point

$$x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$$

and consider the line

$$L_i := \{(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0) \in \mathbb{R}^n \mid x_i \in \mathbb{R}\} .$$

This line is parallel to the x_i-axis and goes through x^0. Then the set

$$\mathcal{C}_i(x^0) := \{(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0, f(\ldots x_{i-1}^0, x_i, x_{i+1}^0, \ldots)) \mid x_i \in \mathbb{R}\}$$

is a curve over the line L_i. It is the graph of the function

$$\begin{array}{rccl} \varphi_i : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x_i & \longmapsto & f(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0) . \end{array}$$

Partial derivatives

$$
\begin{array}{rccl}
\varphi_i : & \mathbb{R} & \longrightarrow & \mathbb{R} \\
 & x_i & \longmapsto & f(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0) \,.
\end{array}
$$

We consider the derivative of φ_i with respect to the variable x_i in the point x^0. This is called the partial derivative of f in x^0 with respect to x_i and written

$$
f_i(x^0) \qquad \text{or} \qquad \frac{\partial f}{\partial x_i}(x^0) \,.
$$

It is defined to be the limit

$$
f_i(x^0) := \lim_{\Delta x \to 0} \frac{f(x_1^0, \ldots, \; x_i^0 + \Delta x \;, \ldots, x_n^0) - f(x_1^0, \ldots, x_i^0, \ldots, x_n^0)}{\Delta x} \,.
$$

Partial derivatives

The tangent at $\mathcal{C}_i(x^0)$ in $p = \left(x^0, f(x^0)\right)$ is given by

$$\left\{ \left(x^0, f(x^0)\right) + \frac{\partial}{\partial x_i} f(x^0)(x_i - x_i^0) \;\middle|\; x_i \in \mathbb{R} \right\} .$$

The tangents $\mathcal{C}_i(x^0)$, $i = 1, \ldots, n$, span the tangent vector space

$$T_p S$$

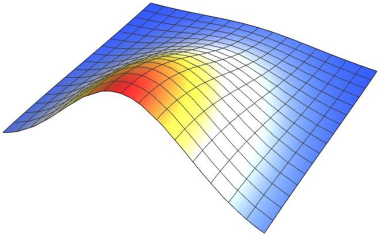
at the surface S in p.

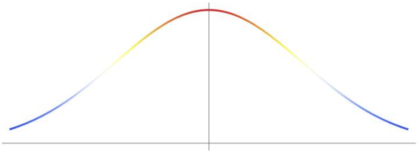
Partial derivatives

We cut the graph of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \,. \end{array}$$

along the plane $y = 0$.





Example

Determine the partial derivatives. We choose the point $p_0 = (x_0, y_0)$. The line $L_1 := \{(x, y_0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is parallel to the x-axis and passes through p_0. Then the set

$$\mathcal{C}_1(p_0) := \{(x, y_0, f(x, y_0)) \mid x \in \mathbb{R}\}$$

is a curve over the line L_1. It is the graph of the function

$$\begin{array}{rccc} \varphi_1 : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x, y_0) = e^{-(x^2+y_0^2)} . \end{array}$$

The partial derivative of f in $p_0 = (x_0, y_0)$ with respect to x is

$$f_x(p_0) = \frac{\partial}{\partial x} f(p_0) = -2x\, e^{-(x^2+y^2)}\Big|_{(x_0,y_0)} = -2x_0\, e^{-(x_0^2+y_0^2)} .$$

Example

With an analogous argument we see that the partial derivative of f in $p_0 = (x_0, y_0)$ with respect to y is

$$f_y(p_0) = \frac{\partial}{\partial y} f(p_0) = -2y\, e^{-(x^2+y^2)}\Big|_{(x_0,y_0)} = -2y_0\, e^{-(x_0^2+y_0^2)}\,.$$

The gradient

We assume that all partial derivatives $\frac{\partial}{\partial x_i} f$, $i = 1, \dots, n$ of the function

$$
\begin{array}{rccc}
f: & D & \longrightarrow & \mathbb{R} \\
 & x & \longmapsto & f(x)
\end{array}
$$

exist and that they are continuous. Then the vector

$$\nabla f(x^0) := \left(\frac{\partial}{\partial x_1} f(x^0), \dots, \frac{\partial}{\partial x_n} f(x^0) \right)$$

is defined and called the gradient of f in x^0.

Example

Compute the gradient of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \, . \end{array}$$

in (x, y).

Example

Compute the gradient of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \, . \end{array}$$

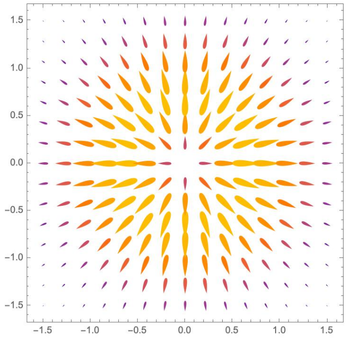
in (x, y) is

$$\begin{aligned} \nabla f(x,y) &= \left(\frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y) \right) = (f_x(x,y), f_y(x,y)) \\ &= \left(-2x\, e^{-(x^2+y^2)}, -2y\, e^{-(x^2+y^2)} \right) \\ &= 2\, e^{-(x^2+y^2)} (-x, -y) \\ &= \frac{2}{e^{(x^2+y^2)}} (-x, -y) \end{aligned}$$

Example

$$\nabla f(x,y) = \frac{2}{e^{(x^2+y^2)}}(-x,-y)$$

In any point $(x,y) \in \mathbb{R}^2$ the gradient points to the origin $(0,0)$ and its length depends on the norm $\sqrt{x^2+y^2}$ of the vector (x,y), hence on the distance of (x,y) to the origin.



The gradient: a property

The gradient $\nabla f(x^0)$ is perpendicular to the level set

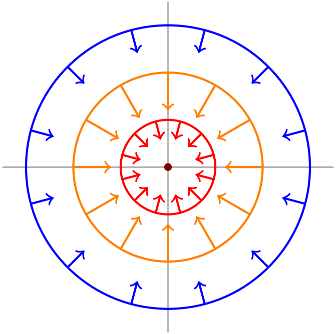
$$f^{-1}\left(f(x^0)\right) := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) = f(x^0)\right\}.$$

Example: Gradient and level sets

The level sets of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \end{array}$$

are circles and the gradient of f in the point (x,y) is parallel to $(-x,-y)$ and points to the origin. As we can see it on the figure, the gradient is perpendicular to the circles.



This week there will an exercise class on Friday!

Have a nice week!