Mathematics
Multivariable functions

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D-ARCH

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Last week

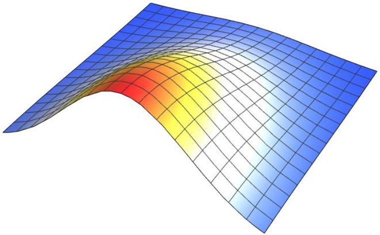
- Multivariable functions
- Partial derivatives
- Gradient: definition

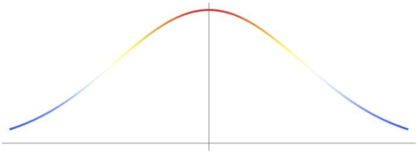
Partial derivatives

We cut the graph of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} . \end{array}$$

along the plane $y = 0$.





Today

- Properties of the gradient
- Total differential
- Chain rule

The gradient

We assume that all partial derivatives $\frac{\partial}{\partial x_i} f$, $i = 1, \ldots, n$ of the function

$$\begin{array}{rccc} f: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

exist and that they are continuous. Then the vector

$$\nabla f(x^0) := \left(\frac{\partial}{\partial x_1} f(x^0), \ldots, \frac{\partial}{\partial x_n} f(x^0) \right)$$

is defined and called the gradient of f in x^0.

Example

Compute the gradient of the function

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \, . \end{array}$$

in (x, y).

Example

The gradient of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)}\,. \end{array}$$

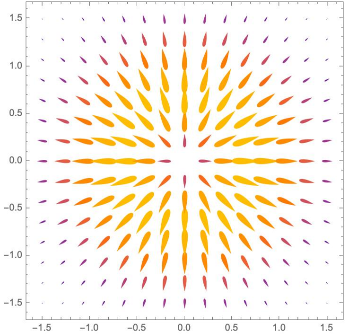
in (x, y) is

$$\begin{aligned} \nabla f(x,y) &= \left(\frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y)\right) = \left(f_x(x,y), f_y(x,y)\right) \\ &= \left(-2x\, e^{-(x^2+y^2)}, -2y\, e^{-(x^2+y^2)}\right) \\ &= 2\, e^{-(x^2+y^2)}(-x,-y) \\ &= \frac{2}{e^{(x^2+y^2)}}(-x,-y) \end{aligned}$$

Example

$$\nabla f(x, y) = \frac{2}{e^{(x^2+y^2)}}(-x, -y)$$

In any point $(x, y) \in \mathbb{R}^2$ the gradient points to the origin $(0, 0)$ and its length depends on the norm $\sqrt{x^2 + y^2}$ of the vector (x, y), hence on the distance of (x, y) to the origin.



The gradient: a property

The gradient $\nabla f(x^0)$ is perpendicular to the level set

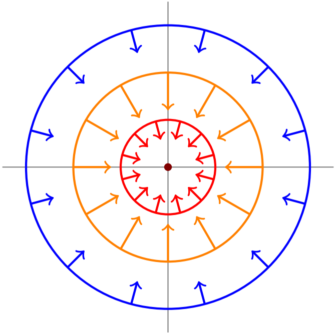
$$f^{-1}\left(f(x^0)\right) := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) = f(x^0)\right\}.$$

Example: Gradient and level sets

The level sets of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \end{array}$$

are circles and the gradient of f in the point (x,y) is parallel to $(-x,-y)$ and points to the origin. As we can see it on the figure, the gradient is perpendicular to the circles.



The directional derivative

By definition, the partial derivative with respect to x_i of a function

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

in x^0 indicates how the value of the function changes when only the x_i-coordinate in x^0 changes, hence when x^0 is moved along a line that is parallel to the x_i-axis.

The directional derivative

How does the value of the function change when we move x^0 along any $v \in \mathbb{R}^n$?

The answer is given by the directional derivative.

The directional derivative

The directional derivative of a scalar function

$$\begin{aligned} f : \quad \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) \end{aligned}$$

along a vector $v \in \mathbb{R}^n$ is the function $\nabla_v f : \mathbb{R}^n \to \mathbb{R}$ defined by the limit

$$\nabla_v f(x) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h|v|} .$$

The division by $|v|$ ensures that the result only depends on the direction of v.

The directional derivative

If the function f is differentiable at x^0, then the directional derivative exists along any vector v and

$$\nabla_v f(x^0) = \nabla f(x^0) \cdot \frac{v}{|v|}$$

where ∇f denotes the gradient of f and "$\cdot$" is the scalar product (dot product) of vectors. The division by $|v|$ ensures that the result does not depend on the magnitude of v.

The directional derivative

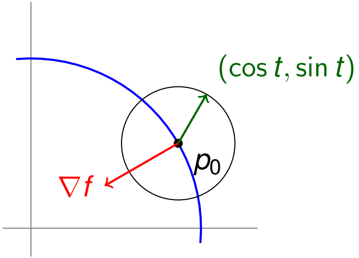
The directional derivative of the function

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y) := e^{-(x^2+y^2)} \end{array}$$

along $v = (\cos t, \sin t)$ equals

$$\nabla_v f(x,y) = \nabla f(x,y) \cdot v$$

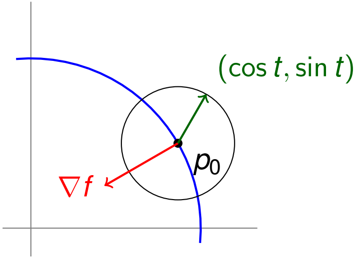
since $|v| = \sqrt{\cos^2 t + \sin^2 t} = 1$.



The directional derivative

$$\begin{aligned}\nabla_v f(x,y) &= \nabla f(x,y)\cdot v\\ &= \left(-2x\,e^{-(x^2+y^2)}, -2y\,e^{-(x^2+y^2)}\right)\cdot(\cos t,\sin t)\\ &= \frac{-2(x\cos t + y\sin t)}{e^{(x^2+y^2)}}\end{aligned}$$

since $|v| = \sqrt{\cos^2 t + \sin^2 t} = 1$.



Direction of the gradient

The gradient ∇f is also denoted $\mathtt{grad}(f)$. It indicates the direction of maximal slope.

Indeed, given x^0 and f, the directional derivative

$$\begin{aligned}\nabla_v f(x^0) &= \nabla f(x^0) \cdot \frac{v}{|v|} = \left|\nabla f(x^0)\right| \left|\frac{v}{|v|}\right| \cos\alpha \\ &= \left|\nabla f(x^0)\right| \cos\alpha\end{aligned}$$

is maximal if the angle α between $\nabla f(x^0)$ and v is 0, i.e., if v has the same direction as the gradient.

The total differential

The total differential of a function $f : \mathbb{R}^3 \to \mathbb{R}$ in 3 variables in (x_0, y_0, z_0) is defined to be

$$df = f_x(x_0, y_0, z_0)\, dx + f_y(x_0, y_0, z_0)\, dy + f_z(x_0, y_0, z_0)\, dz\,.$$

It is a scalar product

$$df = \nabla_V f(x^0, y^0, z^0) \cdot (dx, dy, dz)\,.$$

The total differential of a function in 2 variables in (x_0, y_0) is

$$\begin{aligned} df &= f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy \\ &= \nabla_V f(x^0, y^0) \cdot (dx, dy)\,. \end{aligned}$$

A chain rule for partial derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Consider the composition

$$f\big(x(s,t), y(s,t)\big)$$

for differentiable functions $x, y : \mathbb{R}^2 \to \mathbb{R}$. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

A chain rule for partial derivatives

For the special case $x(t)$, $y(t)$ we have

$$\frac{df}{dt}\big(x(t), y(t)\big) = \frac{\partial f}{\partial x}\big(x(t), y(t)\big) \cdot x'(t) + \frac{\partial f}{\partial y}\big(x(t), y(t)\big) \cdot y'(t)\,.$$

This week there will an exercise class tomorrow!

See you tomorrow!