Mathematics

IBS

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Introduction: an example

We start with a set with $N(t)$ elements at the time t. They can be split into three groups S, I and R with the number of elements $S(t)$, $I(t)$ and $R(t)$.

Hence at any time $t > 0$ we have

$$S(t) + I(t) + R(t) = N(t)$$

We choose $N(t) = N$ to be constant.

Introduction: an example

- An S-element may become I and
- every I-element becomes R.
- Each R-element stays R.

Hence

$$S \xrightarrow{\lambda} I \xrightarrow{\gamma} R$$

Here λ and γ are the rates of change, i.e. the number of individuals per time unit that change the group.

Examples can be found in chemical reactions.

Differential equation

A system of differential equations defines the partition of N, hence the functions $S(t)$, $I(t)$ and $R(t)$.

$$\frac{dS}{dt} = -\lambda S$$

$$\frac{dI}{dt} = \lambda S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

In a few weeks you will be able to solve the system for constant λ and γ.

Differential equation

Unfortunately λ is not a constant but it depends on I

$$\lambda = \beta \frac{I}{N}$$

and we have

$$\frac{dS}{dt} = -\beta \frac{S\,I}{N}$$

$$\frac{dI}{dt} = \beta \frac{S\,I}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

S I R-model

This is the S I R-model: a model for an epidemic within a population.

Susceptible

Infected

Removed

Removed individuals may also be called **R**esistent.
Assumptions:

- Immediately after having been infected, a person is infectious.
- After the recovery a person is immune.

S I R-model

In fact β can be written as a product

$$\beta = q \cdot \kappa \,,$$

where

- κ is the rate of contacts and
- q is the probability of infection in case of contact to an infectious person.

The home-office and remote-teaching reduce κ. Our masks reduce q.

S I R-model

The proportion of infected persons amongst the whole population is $\frac{I}{N}$. It is the probability that a given person is infected.

The force of infection is

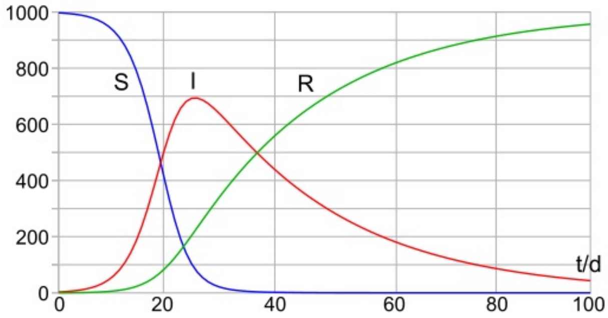
$$\lambda = \beta \frac{I}{N} .$$

The basic reproduction number is

$$R_0 = \frac{\beta}{\gamma} .$$

S I R-model

The solution with $N = 1000$, $S(0) = 997$, $I(0) = 3$, $R(0) = 0$, $\beta = 0.4$ and $\gamma = 0.04$. The time unit is a day.



Graph by Klaus-Dieter Keller, `https://de.wikipedia.org/wiki/SIR-Modell`