Mathematics

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October 30, 2023

Last week

- SIR model
- What is an ordinary differential equation?

Definition

A differential equation is an equation

$$F\big(x, y(x), y'(x), \ldots, y^{(n)}(x)\big) = 0$$

relating the independent variable x, the unknown function

$$y : \mathbb{R} \longrightarrow \mathbb{R}\,.$$

and its derivatives. Here $y^{(i)}$ denotes the ith derivative of y.

The order of the differential equation is the order of the highest derivative that appears in the relation F.

Definition ODE – PDE

The differential equation is called ordinary if y is a function of a single variable.

If $y : \mathbb{R}^n \to \mathbb{R}$ is a function in more than one variable, then the differential equation is called a partial differential equation.
We will study ordinary differential equations.

Today

- First order ODEs
- Separation of variables

First order ODEs

A first order differential equation has the following general form:

$$y'(x) = f\bigl(x, y(x)\bigr) ,$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathrm{dom}(f) = D \subseteq \mathbb{R}^2$.

Differential equation

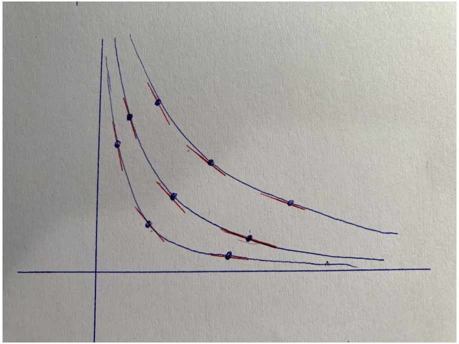
1. The solutions of a differential equation $y'(x) = f(x, y(x))$ form a one-parameter family of curves $y_c : x \to y_c(x)$.
2. The initial value problem (IVP)

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $x \mapsto y(x)$.

First order ODEs

In each point $(x, y) \in \mathbb{R}^2$ the equation $y' = f(x, y)$ defines the slope of the tangent to the graph of the solution of the first-order ODE.



First order ODEs

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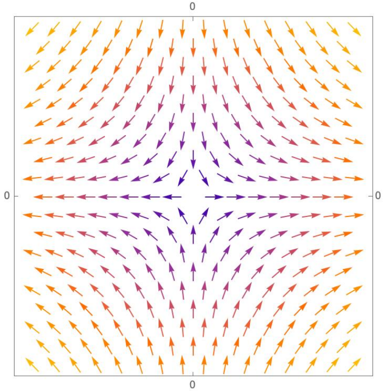
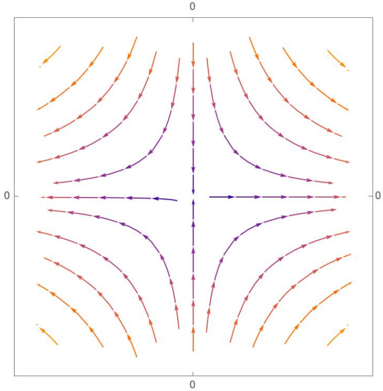


First order ODEs

Each curve in the family of curves is a solution of the differential equation.
For a given initial value $y(a) = b$, the solution is unique and passes through the point (a, b).

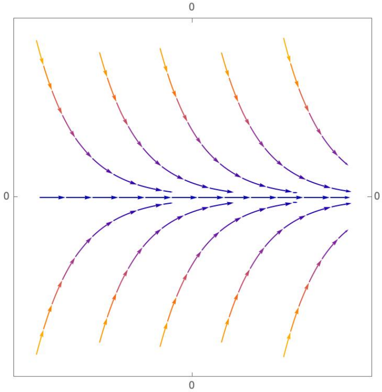


Separation of variables: first example

We solve the ODE

$$\frac{dS}{dt} = -\lambda S$$

for a given constant $\lambda \in \mathbb{R}$.



$\lambda = 2$

Separation of variables: first example

We solve the ODE

$$\frac{dS}{dt} = -\lambda S$$

for a given constant $\lambda \in \mathbb{R}$.
This is done by the separation of variables. We therefore first write

$$dS = -\lambda S \, dt$$

and then

$$\frac{1}{S} \, dS = -\lambda \, dt$$

Note that on the left-hand side we have only the unknown function S and on the right-hand side only the variable t.

Separation of variables: first example

We integrate $\frac{1}{S}\, dS = -\lambda\, dt$

$$\int \frac{1}{S}\, dS = \int -\lambda\, dt$$

$$\ln |S| = -\lambda t + c$$

$$S = e^{-\lambda t + c}$$

$$S = e^{c} \cdot e^{-\lambda t}$$

and get

$$S(t) = C\, e^{-\lambda t}\,,$$

where $C \in \mathbb{R}$ is a constant.

Separation of variables: second example

The differential equation

$$y' = \frac{-x}{y}$$

for the function $y(x)$ can be solved by separation of variables.

Separation of variables: second example

In order to solve the differential equation

$$y' = \frac{-x}{y}$$

we write

$$\frac{dy}{dx} = \frac{-x}{y}\,.$$

Hence

$$dy = \frac{-x}{y}dx$$

and

$$y\,dy = -x\,dx$$

Separation of variables: second example

We integrate both sides of $y\,dy = -x\,dx$

$$\int y\,dy = \int -x\,dx$$

The solution satisfies

$$y^2 = -x^2 + c^2$$

i.e.

$$y = \sqrt{c^2 - x^2}$$

The constant c^2 is defined by the initial value (x_0, y_0).
If $(x_0, y_0) = (3, 4)$, then

$$4^2 = -3^2 + 25 \quad \Rightarrow \quad c^2 = 25\,.$$

Another example

The differential equation

$$y' = -x\,y$$

for the function $y(x)$ can be solved as follows. We write

$$\frac{dy}{dx} = -x\,y$$

Hence

$$dy = -x\,y\,dx$$

and

$$\frac{1}{y}\,dy = -x\,dx$$

Another example

We integrate both sides of $\frac{1}{y}\,dy = -x\,dx$

$$\int \frac{1}{y}\,dy = \int -x\,dx$$

The solution satisfies

$$\ln|y| = -\frac{1}{2}x^2 + c$$

i.e.

$$\begin{aligned} y &= e^{-\frac{1}{2}x^2 + c} = e^{c}\,e^{-\frac{1}{2}x^2} \\ &= C\,e^{-\frac{1}{2}x^2} \end{aligned}$$

Substitutions

Sometimes the differential equation can be solved only after a substitution.

We consider two types:

- $y' = f(ax + by + c)$ and
- $y' = f\left(\frac{y}{x}\right)$

Substitutions

We solve the initial value problem

$$\begin{cases} y' + y = 1 + x, & x > 0 \\ y(0) = 2\,. \end{cases}$$

The differential equation is

$$y' = x - y + 1\,.$$

We define

$$u = x - y + 1$$

and have the derivative with respect to x

$$u' = 1 - y'$$

i.e.

$$y' = 1 - u'\,.$$

Substitutions

We substitute $y' = 1 - u'$ and $y = x - u + 1$ in the differential equation

$$y' = x - y + 1\,.$$

and get

$$\underbrace{1 - u'}_{y'} = x - (\underbrace{x - u + 1}_{y}) + 1 = u\,.$$

The new differential equation

$$u' = 1 - u$$

is separable.

Substitutions

Solve

$$\frac{du}{dx} = 1 - u$$

$$\frac{1}{1-u}\,du = dx$$

by integration

$$\int \frac{1}{1-u}\,du = \int dx$$

$$-\ln(1-u) = x + c_1$$

$$\ln(-x+y) = -x - c_1$$

$$-x + y = e^{-x-c_1}$$

$$y = x + Ce^{-x}\,.$$

Substitutions

For the differential equation

$$y' = f\left(\frac{y}{x}\right)$$

we choose the unknown function

$$u = \frac{y(x)}{x} \quad \Longleftrightarrow \quad x\,u = y\,.$$

Then

$$y' = (u \cdot x)' = u' \cdot x + u$$

and the differential equation for u is

$$u' \cdot x + u = f(u) \quad \Longleftrightarrow \quad u' = \frac{f(u) - u}{x} = \frac{du}{dx}\,.$$

Hence

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

Substitutions

We solve

$$\frac{du}{dx} = \frac{f(u) - u}{x}$$

by integration of

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Hence

$$\underbrace{\int \frac{du}{f(u) - u}}_{I(u)} = \int \frac{dx}{x} = \ln|x| + \mathrm{const} = \ln(c \cdot |x|).$$

Sometimes it is possible to solve

$$I\left(\frac{y}{x}\right) = \ln(c \cdot |x|)$$

with respect to y.

There will be an exercise class on Friday!

See you tomorrow!