Mathematics

Cornelia Busch

ETH Zürich

October 31, 2023

Linear ODE

A linear differential equation is defined by a linear polynomial in the unknown function $y(x)$ and its derivatives $y^{(i)}(x)$.

This is an equation of the form

$$p_n(x)\,y^{(n)}(x) + \ldots + p_2(x)\,y''(x) + p_1(x)\,y'(x) + p_0(x)\,y(x) = q(x)$$

where $p_0(x), \ldots, p_n(x)$ and $q(x)$ are arbitrary differentiable functions. They do not need to be linear.

Linear inhomogeneous ODE

The differential equation is called inhomogeneous if $q(x) \neq 0$

$$p_n(x)\, y^{(n)}(x) + \ldots + p_2(x)\, y''(x) + p_1(x)\, y'(x) + p_0(x)\, y(x) = q(x)$$

and it is called homogeneous if $q(x) = 0$, i.e. if

$$p_n(x)\, y^{(n)}(x) + \ldots + p_2(x)\, y''(x) + p_1(x)\, y'(x) + p_0(x)\, y(x) = 0$$

Given an inhomogeneous linear differential equation, we get the corresponding homogeneous linear differential equation by replacing the function $q(x)$ with the zero function.

An important property

Let $y_1(x)$ and $y_2(x)$ be two different solutions of the inhomogeneous first order linear differential equation

$$p_1(x)\, y'(x) + p_0(x)\, y(x) = q(x)$$

Then

$$p_1(x)\, y_1'(x) + p_0(x)\, y_1(x) = q(x)$$
$$p_1(x)\, y_2'(x) + p_0(x)\, y_2(x) = q(x)$$

An important property

Hence the difference $(y_1 - y_2)(x)$ satisfies

$$\begin{aligned} p_1(x)\,(y_1 - y_2)'(x) + p_0(x)\,(y_1 - y_2)(x) \\ &= p_1(x)\,y_1'(x) - p_1(x)\,y_2'(x) + p_0(x)\,y_1(x) - p_0(x)\,y_2(x) \\ &= \underbrace{p_1(x)\,y_1'(x) + p_0(x)\,y_1(x)}_{=q(x)} - \underbrace{(p_1(x)\,y_2'(x) + p_0(x)\,y_2(x))}_{=q(x)} \\ &= 0 \end{aligned}$$

This shows that the difference of two solutions of the inhomogeneous linear ODE is a solution of the corresponding homogeneous linear ODE.

This is true for any order by a similar argument than the one for the first order.

An important property

The following is true for inhomogeneous linear differential equations.

the general solution of an inhomogeneous linear differential equation

$$=$$

the general solution of the corresponding homogeneous linear differential equation

$$+$$

the particular solution of the inhomogeneous linear differential equation.

First order linear ODEs

The method to solve an inhomogeneous linear ODE is the following.

1) First determine the general solution y_h of the corresponding homogeneous ODE.
2) Then find a particular solution y_p of the inhomogeneous ODE.
3) The general solution of the inhomogeneous linear ODE is

$$y = y_h + y_p$$

First order linear ODEs

In order to solve

$$y'(x) + p(x)y(x) = q(x)\,.$$

First we determine the general solution y_h of

$$y'(x) + p(x)y(x) = 0\,.$$

We then determine a particular solution y_p of

$$y'(x) + p(x)y(x) = q(x)\,.$$

The general solution of our differential equation is

$$y(x) = y_p(x) + y_h(x)$$

First order linear ODE: example 1

Find the general solution of the following differential equation.

$$y' - 2xy = x$$

First order linear ODE: example 1

The general solution of the corresponding homogeneous differential equation

$$y' - 2xy = 0$$

is found by

$$\frac{dy}{dx} = 2xy$$

$$\frac{1}{y}dy = 2x\,dx$$

$$\int \frac{1}{y}dy = \int 2xdx$$

$$\ln|y| = x^2 + c$$

$$y = e^{x^2+c}$$

First order linear ODE: example 1

The general solution of the homogeneous ODE is

$$y_h(x) = C\, e^{x^2}\,.$$

We find the particular solution of the inhomogeneous first order linear ODE by the method of variation of constants.

We guess

$$y_p(x) = C(x)\, e^{x^2}\,.$$

We have to find the function $C(x)$.

First order linear ODE: example 1

In order to determine $C(x)$ we plug our guess $y_p(x) = C(x)\, e^{x^2}$ into the ODE

$$y' - 2xy = x$$

With

$$y_p'(x) = C'(x)\, e^{x^2} + C(x) \cdot 2x\, e^{x^2}$$

we get

$$\begin{aligned} y_p'(x) - 2xy_p(x) &= C'(x)\, e^{x^2} + 2x\, C(x)\, e^{x^2} - 2x\, C(x)\, e^{x^2} \\ &= C'(x)\, e^{x^2} \end{aligned}$$

Now we solve

$$C'(x)\, e^{x^2} = x$$

First order linear ODE: example 1

Now we solve

$$C'(x)\, e^{x^2} = x$$

and get

$$\begin{aligned} C'(x) &= xe^{-x^2} \\ C(x) &= \int xe^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C_1 \,. \end{aligned}$$

Hence

$$y_p(x) = C(x) e^{x^2} = -\frac{1}{2} + C_1 \, e^{x^2}$$

and the general solution is

$$y(x) = y_p(x) + y_h(x) = -\frac{1}{2} + Ce^{x^2} \,.$$

First order linear ODE: example 2

Find the maximal solution of the initial value problem

$$y' - y \cdot \tan x = \cos^2 x \quad \text{with} \quad y(0) = 1\,.$$

First order linear ODE: example 2

The general solution of the corresponding homogeneous differential equation

$$y' - y \cdot \tan x = 0$$

is found by

$$\frac{dy}{dx} = y \cdot \tan x$$

$$\int \frac{1}{y} dy = \int \tan x \, dx$$

$$\ln |y| = -\ln |\cos x| + c$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $(x_0 = 0)$.

First order linear ODE: example 2

The general solution of the homogeneous ODE is

$$y_h(x) = C \cdot \frac{1}{|\cos x|} \,.$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We guess

$$y_p(x) = C(x)\,\frac{1}{\cos x} \,.$$

We have to find the function $C(x)$.

First order linear ODE: example 2

In order to determine $C(x)$ we plug our guess $y_p(x) = C(x)\,\frac{1}{\cos x}$ into the ODE

$$y' - y \cdot \tan x = \cos^2 x$$

With

$$y_p'(x) = C'(x)\,\frac{1}{\cos x} + C(x) \cdot \frac{\tan x}{\cos x}$$

we get

$$\begin{aligned} y_p'(x) - y_p(x) \cdot \tan x &= C'(x)\,\frac{1}{\cos x} + C(x) \cdot \frac{\tan x}{\cos x} - C(x) \cdot \frac{\tan x}{\cos x} \\ &= C'(x)\,\frac{1}{\cos x} \end{aligned}$$

First order linear ODE: example 2

Now we solve

$$C'(x)\,\frac{1}{\cos x} = \cos^2 x$$

and get

$$C(x) = \int \cos^3(x)dx$$
$$= \sin x - \frac{1}{3}\sin^3 x\,.$$

Hence

$$y_p(x) = \frac{\sin x - \frac{1}{3}\sin^3 x}{\cos x} \quad \text{with } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

First order linear ODE: example 2

The general solution is

$$y(x) = \frac{C + \sin x - \frac{1}{3}\sin^3 x}{\cos x} \quad \text{with } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

With the condition $y(0) = 1$ we get $C = 1$.

$$y(x) = \frac{1 + \sin x - \frac{1}{3}\sin^3 x}{\cos x} \quad \text{with } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

There will be an exercise class on Friday!

Have a nice week!