Mathematics

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Program

Last week

- Separation of variables.
- First order linear differential equations: variation of constants.
- Inhomogeneous linear differential equations.

Today

- Linear differential equations with constant coefficients.

Linear ODEs of order n with constant coefficients

A linear differential equation of order n is

$$x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_1(t)\dot{x}(t) + a_0(t)x(t) = r(t)\,.$$

If $a_{n-1}(t), \ldots, a_0(t)$ are constant (don't depend on t), then the differential equation has constant coefficients.

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \ldots + a_1\dot{x}(t) + a_0x(t) = r(t)\,.$$

Linear inhomogeneous ODE

We use the following method.

1) First determine the general solution x_h of the corresponding homogeneous ODE.
2) Then find a particular solution x_p of the inhomogeneous ODE.
3) The general solution of the inhomogeneous linear ODE is

$$x = x_h + x_p$$

General solution of the homogeneous ODE

We guess

$$x(t) = e^{\lambda t}\,, \quad \lambda \in \mathbb{R}$$

and plug this function in the homogeneous equation

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \ldots + a_1\dot{x}(t) + a_0 x(t) = 0\,.$$

With

$$x^{(i)}(t) = \lambda^i e^{\lambda t}$$

we get

$$\lambda^n e^{\lambda t} + a_{n-1}\lambda^{n-1}e^{\lambda t} + \ldots a_1\lambda e^{\lambda t} + a_0 e^{\lambda t} = 0$$

and

$$e^{\lambda t}(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0) = 0\,.$$

General solution of the homogeneous ODE

We consider the equation

$$e^{\lambda t}(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0) = 0\,.$$

Since $e^{\lambda t} \neq 0$, it is equivalent to

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0\,.$$

We call

$$Q(\lambda) := \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

the characteristic polynomial of the differential equation.

$$x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1\dot{x} + a_0 x = 0\,.$$

General solution of the homogeneous ODE

There are different cases:

1. $Q(\lambda)$ has n different real zeros.
2. $Q(\lambda)$ has zeros with multiplicity > 1.
3. $Q(\lambda)$ has complex zeros.

General solution of the homogeneous ODE

If $Q(\lambda) = 0$ has n different real zeros $\lambda_1, \ldots, \lambda_n$, then the differential equation has n linear independent solutions:

$$e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$$

The general solution is

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}$$

with n constants $c_1, \ldots, c_n$.

General solution of the homogeneous ODE

If λ_k is a zero with multiplicity p, then

$$e^{\lambda_k t}, te^{\lambda_k t}, t^2 e^{\lambda_k t}, \ldots, t^{p-1} e^{\lambda_k t}$$

are the p solutions corresponding to λ_k.

Their contribution to the general equation is

$$C_0 e^{\lambda_k t} + C_1 te^{\lambda_k t} + C_2 t^2 e^{\lambda_k t} + \cdots + C_{p-1} t^{p-1} e^{\lambda_k t}$$

General solution of the homogeneous ODE

By the previous scheme the pair of complex conjugate zeros

$$\lambda = \alpha \pm i\beta$$

would contribute with the complex solutions

$$\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big) \\ e^{(\alpha-i\beta)t} &= e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big) \end{aligned}$$

but we want real functions. The functions

$$e^{\alpha t}\cos(\beta t) \quad \text{and} \quad e^{\alpha t}\sin(\beta t)\,.$$

are the real solutions. These are linearly independent.

General solution of the homogeneous ODE

If the multiplicity of the complex zero is $p > 1$ we have the solutions

$$\begin{array}{cccc} e^{\alpha t}\cos(\beta t)\,, & t\,e^{\alpha t}\cos(\beta t)\,, & \ldots, & t^{p-1}\,e^{\alpha t}\cos(\beta t)\,, \\ e^{\alpha t}\sin(\beta t)\,, & t\,e^{\alpha t}\sin(\beta t)\,, & \ldots, & t^{p-1}\,e^{\alpha t}\sin(\beta t)\,. \end{array}$$

Particular solution of the inhomogeneous ODE

To find the particular solution of the inhomogeneous ODE

$$x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_1(t)\dot{x}(t) + a_0(t)x(t) = r(t)$$

we make a guess that depends on $r(t)$.

See the examples!

Example 1

Determine all the solutions $y(x)$ of the differential equation

$$y'' - y' - 6y = e^{-x} .$$

that are bounded on the interval $[0, \infty[$ and satisfy $y(0) = 0$.

Example 1

We first consider the homogeneous equation

$$y'' - y' - 6y = 0\,.$$

The zeros of the characteristic polynomial

$$\lambda^2 - \lambda - 6 = 0$$

are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} = \begin{cases} 3 \\ -2 \end{cases}\,.$$

The general solution of the homogeneous equation is

$$y_h(x) = c_1 e^{3x} + c_2 e^{-2x}\,.$$

Example 1

The guess for the special solution of the inhomogeneous equation

$$y'' - y' - 6y = e^{-x}$$

is

$$y(x) = c\,e^{-x}\,.$$

Then $y' = -ce^{-x}$, $y'' = ce^{-x}$ and

$$\begin{aligned} y'' - y' - 6y &= ce^{-x} + ce^{-x} - 6ce^{-x} \\ &= -4ce^{-x} \stackrel{!}{=} e^{-x}\,. \end{aligned}$$

The particular solution is

$$y_p(x) = -\frac{1}{4}e^{-x}$$

Example 1

The general solution is

$$y(x) = y_h(x) + y_p(x) = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{4} e^{-x} \,.$$

We study the solution on the interval $[0, \infty[$:

$$\begin{array}{ll} e^{\alpha x} > 0 & \text{for } \alpha \in \mathbb{R},\ x \in \mathbb{R} \\ e^0 = 1 & \\ \lim_{x\to\infty} e^{\alpha x} = \infty & \text{for } \alpha \in \mathbb{R},\ \alpha > 0 \\ \lim_{x\to\infty} e^{\alpha x} = 0 & \text{for } \alpha \in \mathbb{R},\ \alpha < 0 \end{array}$$

therefore $y(x)$ is bounded if and only if $c_1 = 0$. The condition $y(0) = 0$ yields the solution

$$y(x) = \frac{1}{4} e^{-2x} - \frac{1}{4} e^{-x} \,.$$

Example 2

Determine all the solutions $y(x)$ of the differential equation

$$y'' - y' - 2y = x^2$$

that satisfy $y(0) = 0$ and $y'(0) = \frac{1}{2}$.

Example 2

We first consider the homogeneous equation

$$y'' - y' - 2y = 0\,.$$

The zeros of the characteristic polynomial

$$\lambda^2 - \lambda - 2 = 0$$

are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}.$$

The general solution of the homogeneous equation is

$$y_h(x) = c_1 e^{2x} + c_2 e^{-x}\,.$$

Example 2

The guess for the special solution of the inhomogeneous equation

$$y'' - y' - 2y = x^2$$

is

$$y(x) = Ax^2 + Bx + C\,.$$

Then $y' = 2Ax + B$, $y'' = 2A$ and

$$\begin{aligned} y'' - y' - 2y &= 2A - (2Ax + B) - 2(Ax^2 + Bx + C) \\ &= -2Ax^2 - (2A + 2B)x + 2A - B - 2C \\ &\overset{!}{=} x^2\,. \end{aligned}$$

Example 2

Comparing the coefficients of both sides of

$$-2Ax^2 - (2A + 2B)x + 2A - B - 2C = x^2$$

we get the following system of equations

$$\begin{aligned} -2A &= 1 \\ -2A - 2B &= 0 \\ 2A - B - 2C &= 0 \end{aligned}$$

We see that

$$A = -\frac{1}{2}, \quad B = \frac{1}{2}, \quad C = -\frac{3}{4}$$

The particular solution is

$$y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}.$$

Example 2

The general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^{2x} + c_2 e^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}. \end{aligned}$$

The conditions $y(0) = 0$ and $y'(0) = \frac{1}{2}$ define the constants c_1 and c_2. Since

$$y'(x) = 2c_1 e^{2x} - c_2 e^{-x} - x + \frac{1}{2},$$

we get

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 - \frac{3}{4} \\ y'(0) = \frac{1}{2} &= 2c_1 - c_2 + \frac{1}{2} \end{aligned}$$

Example 2

The solutions of the system

$$c_1 + c_2 = \frac{3}{4}$$
$$2c_1 - c_2 = 0$$

are

$$c_1 = \frac{1}{4}\,, \quad c_2 = \frac{1}{2}\,.$$

Hence the solution of our initial value problem is

$$y(x) = \frac{1}{4}e^{2x} + \frac{1}{2}e^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}\,.$$

Example 3

Determine all the solutions $y(x)$ of the differential equation

$$y'' - y' - 2y = \cos(x)$$

that satisfy $y(0) = 0$ and $y'(0) = \frac{1}{2}$.

We already know the general solution of the homogeneous equation:

$$y_h(x) = c_1 e^{2x} + c_2 e^{-x} .$$

Example 3

The guess for the special solution of the inhomogeneous equation

$$y'' - y' - 2y = \cos(x)$$

is

$$y(x) = A\cos(x) + B\sin(x)\,.$$

Then

$$\begin{aligned} y' &= -A\sin(x) + B\cos(x) \\ y'' &= -A\cos(x) - B\sin(x) \end{aligned}$$

and

$$\begin{aligned} y'' - y' - 2y &= -A\cos(x) - B\sin(x) - (-A\sin(x) + B\cos(x)) - 2(A\cos(x) + B\sin(x)) \\ &= (-3A - B)\cos(x) + (A - 3B)\sin(x) \\ &\overset{!}{=} \cos(x)\,. \end{aligned}$$

Example 3

Comparing the coefficients of both sides of

$$(-3A - B)\cos(x) + (A - 3B)\sin(x) = \cos(x)$$

we get the following system of equations

$$\begin{aligned} -3A - B &= 1 \\ A - 3B &= 0 \end{aligned}$$

We see that

$$A = -\frac{3}{10}, \quad B = -\frac{1}{10}$$

The particular solution is

$$y_p(x) = -\frac{3}{10}\cos(x) - \frac{1}{10}\sin(x).$$

Example 3

The general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^{2x} + c_2 e^{-x} - \frac{3}{10}\cos(x) - \frac{1}{10}\sin(x)\,. \end{aligned}$$

The conditions $y(0) = 0$ and $y'(0) = \frac{1}{2}$ define the constants c_1 and c_2. Since

$$y'(x) = 2c_1 e^{2x} - c_2 e^{-x} + \frac{3}{10}\sin(x) - \frac{1}{10}\cos(x)\,,$$

we get

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 - \frac{3}{10} \\ y'(0) = \frac{1}{2} &= 2c_1 - c_2 - \frac{1}{10} \end{aligned}$$

Example 3

The solutions of the system

$$c_1 + c_2 = \frac{3}{10}$$
$$2c_1 - c_2 = \frac{6}{10}$$

are

$$c_1 = \frac{3}{10}\,, \qquad c_2 = 0\,.$$

Hence the solution of our initial value problem is

$$y(x) = \frac{3}{10}e^{2x} - \frac{3}{10}\cos(x) - \frac{1}{10}\sin(x)\,.$$

There there will be an exercise class tomorrow!

See you tomorrow!