Mathematics

IBS

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A system of equations

We want to solve

$$\left|\begin{array}{rcrcl} 3x_1 & + & x_2 & = & 2 \\ 5x_1 & + & 2x_2 & = & 3 \end{array}\right.$$

We first eliminate x_1 in the second row and then x_2 in the first row:

$$\left|\begin{array}{rcrcl} 3x_1 & + & x_2 & = & 2 \\ 3x_1 & + & \frac{6}{5}x_2 & = & \frac{9}{5} \end{array}\right.$$

$$\left|\begin{array}{rcrcr} 3x_1 & + & x_2 & = & 2 \\ & & \frac{1}{5}x_2 & = & -\frac{1}{5} \end{array}\right.$$

A system of equations

$$\left| \begin{array}{rcrcr} 3x_1 & + & x_2 & = & 2 \\ & & x_2 & = & -1 \end{array} \right.$$

$$\left| \begin{array}{rcrcr} 3x_1 & & & = & 3 \\ & & x_2 & = & -1 \end{array} \right.$$

Solution

$$\left| \begin{array}{rcr} x_1 & = & 1 \\ x_2 & = & -1\,. \end{array} \right.$$

A linear function

In the example we consider a function

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f(x) \end{array}$$

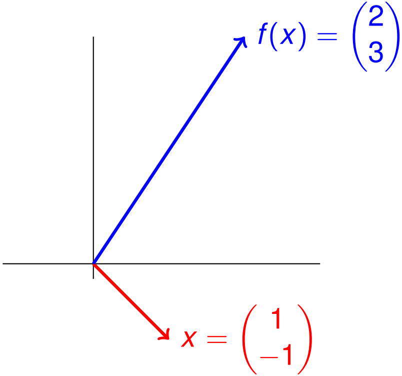
defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix}$$

and search for a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ whose image is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, i.e.

$$f(x) = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} .$$

Picture



$f(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Inverse function

Let f be a function whose domain is the set A and whose codomain is the set B. Then f is invertible if there exists a function g whose domain is the set B and whose image is the set A, with the property

$$f(x) = y \quad \Longleftrightarrow \quad g(y) = x\,.$$

Then

$$g(f(x)) = x \quad \text{and} \quad f(g(y)) = y\,.$$

The inverse g is denoted f^{-1}.

A linear function

The linear function

$$f: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \longmapsto & \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix} = f(x) \end{array}$$

is invertible. Its inverse is the function

$$g: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \longmapsto & \begin{pmatrix} 2x_1 - x_2 \\ -5x_1 + 3x_2 \end{pmatrix} = g(x) \, . \end{array}$$

Solution of a system of equations

The solution of the equation

$$f(x) = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = b$$

is

$$g(b) = g\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution of a system of equations

The system of linear equations

$$f(x) = b$$

has a unique solution if and only if f is invertible. Then

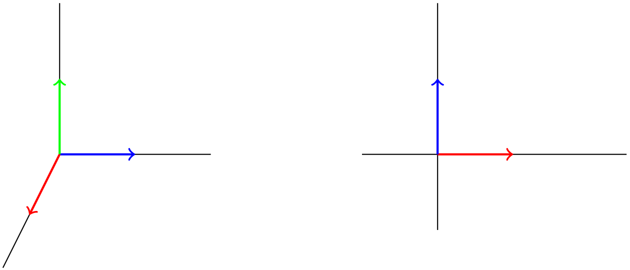
$$x = g(b)\,.$$

Non-invertible linear functions

The nonzero linear mapping

$$
\begin{array}{rccc}
f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\
 & x & \longmapsto & f(x)
\end{array}
$$

is never invertible since it is never injective.



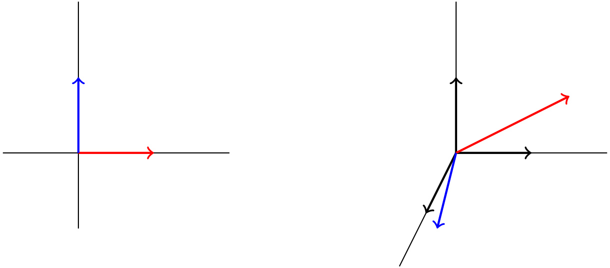
Its image is $\mathbb{R}^2$ or a one-dimensional subspace of $\mathbb{R}^2$, i.e., a line across $(0, 0)$.

Non-invertible linear functions

The nonzero linear mapping

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & x & \longmapsto & f(x) \end{array}$$

is never invertible since it is never surjective.



Its image is a one- ore two-dimensional subspace of $\mathbb{R}^3$, i.e., a line or a plane across $(0, 0, 0)$.

Solution of a system of linear equations

Let

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & x & \longmapsto & f(x) \end{array}$$

and $b \in \mathbb{R}^m$. We consider the equation

$$f(x) = b\,.$$

- If f is bijective, then the equation $f(x) = b$ has a unique solution.
- If f is not injective and $b \in \mathrm{im}(f)$, then $f(x) = b$ has an infinite number of solutions.
- If f is not surjective and $b \notin \mathrm{im}(f)$, then $f(x) = b$ has no solution.

Linear dependency

A set of n vectors $\alpha_1, \ldots \alpha_n \in \mathbb{R}^m$ is called linearly dependent if there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n = 0$$

and there is $i \in \{1, \ldots n\}$ with $\lambda_i \neq 0$ (at least one of the λ_i is nonzero).

The set of vectors $\alpha_1, \ldots \alpha_n \in \mathbb{R}^m$ is called linearly independent if and only if

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n = 0$$

implies

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0\,.$$

Basis

The set

$$< \alpha_1, \dots, \alpha_n > := \left\{ \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \cdots + \lambda_n \alpha_n \,\middle|\, \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}$$

is called the span of $\alpha_1, \dots, \alpha_n$. It is the set of all linear combinations of $\alpha_1, \dots, \alpha_n$.

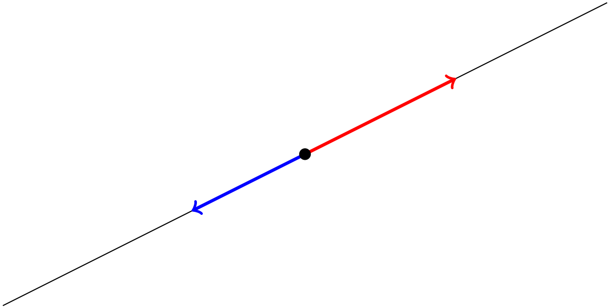
The dimension of $V := < \alpha_1, \dots, \alpha_n >$ is the maximal number m such that there are m linearly independent vectors in V.

Then the m linearly independent vectors are called a basis of V.

Span of $\mathbb{R}$

One vector spans a line. Two vectors on the same line are always linearly dependent.

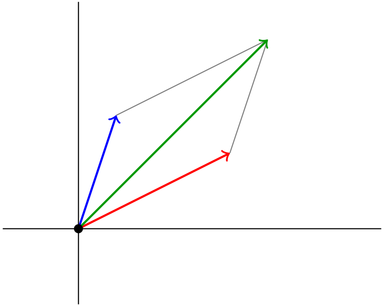
A line is a one-dimensional space.



Span of $\mathbb{R}^2$

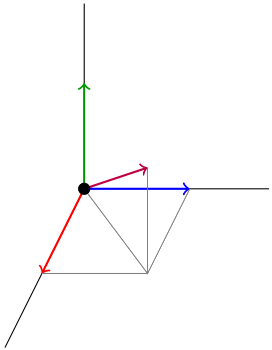
Two linearly independent vectors span a plane. Three vectors in the same plane are always linearly dependent.

A plane is a two-dimensional space.



Span of $\mathbb{R}^3$

Three linearly independent vectors span a 3-dimensional space. Four vectors in a 3-dimensional space are always linearly dependent.



Standard basis

The standard basis of $\mathbb{R}^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the standard basis of $\mathbb{R}^3$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Standard basis

For every vector $x \in \mathbb{R}^2$ there are $x_1, x_2 \in \mathbb{R}$ such that

$$x = x_1\, e_1 + x_2\, e_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and for every vector $y \in \mathbb{R}^3$ there are $y_1, y_2, y_3 \in \mathbb{R}$ such that

$$y = y_1\, e_1 + y_2\, e_2 + y_3\, e_3 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} .$$

Linear mappings

Let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$.
Let $x = x_1 v_1 + \ldots + x_n v_n \in \mathbb{R}^n$, then

$$\begin{aligned} f(x) &= f(x_1 v_1 + \ldots + x_n v_n) \\ &= x_1 f(v_1) + \ldots + x_n f(v_n) \end{aligned}$$

Hence we know the image of any $x \in \mathbb{R}^n$ if we know

$$f(v_1), \ldots, f(v_n).$$

Linear mappings

We define a linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(e_1) = 3e_1 + 5e_2$$
$$f(e_2) = e_1 + 2e_2$$

Then

$$\begin{aligned} f(x_1 e_1 + x_2 e_2) &= x_1 f(e_1) + x_2 f(e_2) \\ &= x_1(3e_1 + 5e_2) + x_2(e_1 + 2e_2) \\ &= 3x_1 e_1 + 5x_1 e_2 + x_2 e_1 + 2x_2 e_2 \\ &= (3x_1 + x_2)e_1 + (5x_1 + 2x_2)e_2 \end{aligned}$$

Hence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix}$$

Linear mappings

We introduce a matrix A such that

$$f(x) = Ax\,.$$

This is the multiplication of the vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with a matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

$$Ax = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix}$$