Mathematics

IBS

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Linear equations

Given $a, b \in \mathbb{R}$. When does the linear equation in x

$$a \cdot x = b$$

have a solution? Is there more than one solution?

Let a and b be rational, real or complex numbers. If it is possible to divide by a, i.e. $a \neq 0$ or a is invertible, then the equation has a unique solution

$$x = a^{-1} \cdot b = \frac{b}{a}\,.$$

Linear equations

If $a = 0$ and $b = 0$ we have the equation

$$0 \cdot x = 0$$

and this is true for any value of x. Therefore the equation $0 \cdot x = 0$ has infinitely many solutions.

If $a = 0$ and $b \neq 0$, then the equation

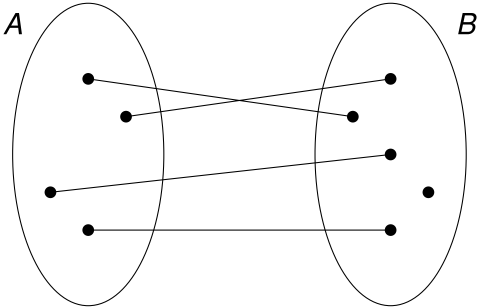
$$0 \cdot x = b \neq 0$$

has no solutions.

Injective, surjective, bijective

A function $f : A \to B$ is said to be injective (or one-to-one) if for all $x_1, x_2 \in A$, whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$. In symbols

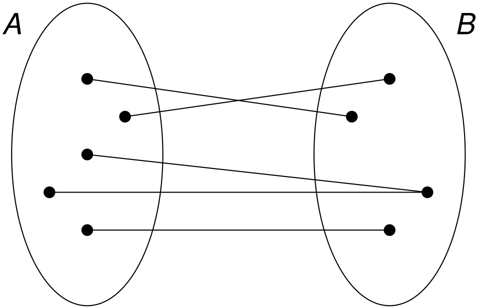
$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2 \,.$$



Injective, surjective, bijective

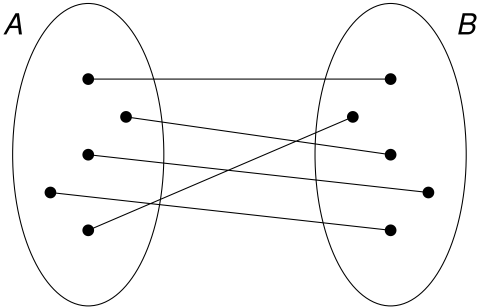
A function $f : A \to B$ is said to be surjective (or onto) if for every $y \in B$, there exists an $x \in A$, such that $f(x) = y$. In symbols

$$\forall y \in B,\ \exists x \in A, \quad f(x) = y\,.$$



Injective, surjective, bijective

A function $f : A \to B$ is said to be bijective if it is injective and surjective.



Linear functions

Let

$$\begin{aligned} f: \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) \end{aligned}$$

be a function mapping the real numbers on real numbers. The function f is called linear if and only if for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{R}$

$$f(\lambda x + y) = \lambda f(x) + f(y)\,.$$

Linear functions

Let $a \in \mathbb{R}$ be a constant. Then the function given by

$$f(x) = ax, \qquad x \in \mathbb{R}$$

is a linear function.

The function

$$g(x) = ax + b, \qquad x \in \mathbb{R}$$

is not linear.
Here $a, b \in \mathbb{R}$ are constants.

Linear functions

Let

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & x & \longmapsto & f(x) \end{array}$$

be a function mapping a vector with n components on vectors with m components (coordinates)

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix} = f(x)\,,$$

where

$$\begin{array}{rccc} f_i: & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f_i(x)\,, \end{array}$$

$i = 1, \ldots, n$, is a function in n variables.

Vector space

The sets $\mathbb{R}^n$ and $\mathbb{R}^m$ are vector spaces.

The natural number n is the dimension of the vector space $\mathbb{R}^n$.

Example of a linear function

We consider the function

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f(x) \end{array}$$

defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 2x_2 \end{pmatrix}$$

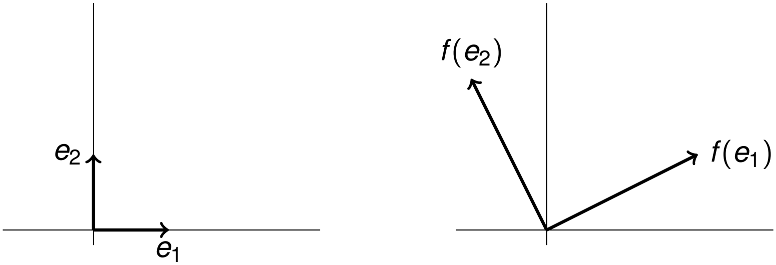
In particular

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 2 \\ 1 \end{pmatrix} = f(e_1) \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 \\ 2 \end{pmatrix} = f(e_2)$$

A picture

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 2 \\ 1 \end{pmatrix} = f(e_1) \qquad \text{and} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 \\ 2 \end{pmatrix} = f(e_2)$$

We represent the vectors e_1, e_2 and their images in different pictures.



This week there will be an exercise class on Friday!

Have a nice week!