Mathematics

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A system of equations

We solve

$$\left| \begin{array}{rcrcr} 3x_1 & + & x_2 & = & 2 \\ 5x_1 & + & 2x_2 & = & 3 \end{array} \right.$$

and get

$$\left| \begin{array}{rcr} x_1 & = & 1 \\ x_2 & = & -1\,. \end{array} \right.$$

Standard basis

The standard basis of $\mathbb{R}^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the standard basis of $\mathbb{R}^3$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Standard basis

For every vector $x \in \mathbb{R}^2$ there are $x_1, x_2 \in \mathbb{R}$ such that

$$x = x_1\, e_1 + x_2\, e_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and for every vector $y \in \mathbb{R}^3$ there are $y_1, y_2, y_3 \in \mathbb{R}$ such that

$$y = y_1\, e_1 + y_2\, e_2 + y_3\, e_3 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} .$$

Linear mappings

Let $v_1, \dots, v_n$ be a basis of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$.
Let $x = x_1 v_1 + \ldots + x_n v_n \in \mathbb{R}^n$, then

$$
\begin{aligned}
f(x) &= f(x_1 v_1 + \ldots + x_n v_n) \\
&= x_1 f(v_1) + \ldots + x_n f(v_n)
\end{aligned}
$$

Hence we know the image of any $x \in \mathbb{R}^n$ if we know

$$
f(v_1), \ldots, f(v_n).
$$

Linear mappings

We define a linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(e_1) = 3e_1 + 5e_2$$
$$f(e_2) = e_1 + 2e_2$$

Then

$$\begin{aligned} f(x_1e_1 + x_2e_2) &= x_1f(e_1) + x_2f(e_2) \\ &= x_1(3e_1 + 5e_2) + x_2(e_1 + 2e_2) \\ &= 3x_1e_1 + 5x_1e_2 + x_2e_1 + 2x_2e_2 \\ &= (3x_1 + x_2)e_1 + (5x_1 + 2x_2)e_2 \end{aligned}$$

Hence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix}$$

Linear mappings

We introduce a matrix A such that

$$f(x) = Ax\,.$$

This is the multiplication of the vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with a matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

$$Ax = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix}$$

Matrix multiplication

The multiplication of a matrix with a vector is done as follows.

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 2x_2 \end{pmatrix}$$

In general

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} .$$

Matrix multiplication

Then

$$Ae_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$Ae_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

Hence

- the first column of A contains the image of the first basis vector e_1 and
- the second column of A contains the image of second basis vector e_2.

Mappings and matrices

We now consider

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & Ax = y \end{array}$$

i.e. A is an $n \times n$-matrix.
Since the images of the basis-vectors of $\mathbb{R}^n$ are the columns of the matrix A, these columns span the image $\mathrm{im}(f)$ of f.

Linear equations

- If the rows of the matrix A are linearly independent, then

$$\mathrm{im}(f) = \mathbb{R}^n .$$

  In this case the mapping f has an inverse.
- If rows of the matrix A are linearly dependent, then

$$\mathrm{im}(f) \subsetneq \mathbb{R}^n$$

  and the dimension of $\mathrm{im}(f)$ is smaller than n. In this case the mapping f has no inverse and there exist vectors $0 \neq c \in \mathbb{R}^n$ that satisfy

$$c \notin \mathrm{im}(f) .$$

Linear equations

In any matrix A the number of linearly independent rows equals the number of linearly independent columns.

This is called the rank of the matrix. It is the dimension of $\text{im}(f)$

Linear equations

We are back to our system of n equations in n variables.

$$f(x) = Ax = b$$

with $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$.

- If the rows of A are linearly independent, then the mapping f has an inverse and the equation has a unique solution x.
- If the rows of A are linearly dependent the mapping has no inverse and we have to consider b.

Linear equations

$$f(x) = Ax = b$$

We assume that f is not invertible.

- If $b \in \mathrm{im}(f)$, the system has an infinite number of solutions.
- If $b \notin \mathrm{im}(f)$, the system has no solution.

Linear equations

For the simplicity of the notation we choose $f : \mathbb{R}^3 \to \mathbb{R}^3$. Given an equation $Ax = b$ with

$$Ax = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b$$

To see wether b is in the image of f, we first write an extended matrix

$$\left| \begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right|$$

Linear equations

If the rank, i.e. the number of linearly independent lines, in

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

equals the rank of

$$\left|\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right|$$

then $b \in \mathrm{im}(f)$ and the system has an infinite number of solutions.

If this is not the case, then $b \notin \mathrm{im}(f)$ and the system has no solution.

Linear equations

We first consider the case of an invertible f. We then have a unique solution.

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

Linear equations

We determine the rank of

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & 1 & -1 & 2 \end{array}\right|$$

We first add multiples of the first row to the second and the third row.

$$\left|\begin{array}{ccr|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right|$$

is equivalent to

$$\left|\begin{array}{ccr|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right|$$

Linear equations

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right|$$

We see that the rank of the first 3 columns in this system equals 3. Hence the system has a unique solution.

Indeed the solution of

$$\begin{array}{rcrcrcr} x_1 & & & - & x_3 & = & 2 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 1 \end{array}$$

is

$$x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Linear equations

How many solutions has the following equation?

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

Linear equations

We determine the rank of

$$\left|\begin{array}{ccr|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & 1 & 0 & 3 \end{array}\right|$$

We first add multiples of the first row to the second and the third row.

$$\left|\begin{array}{ccr|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right|$$

is equivalent to

$$\left|\begin{array}{ccr|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

Linear equations

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

We see that the rank of A equals the rank of this system. Hence the system has an infinite number of solutions.
Indeed the set of solutions of

$$\begin{array}{ccccccc} x_1 & & & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 1 \end{array}$$

is

$$\left\{ x = \begin{pmatrix} 2+\lambda \\ 1-\lambda \\ \lambda \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\}$$

Linear equations

How many solutions has the following equation?

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

Linear equations

We determine the rank of

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & 1 & 0 & 2 \end{array}\right|$$

We first add multiples of the first row to the second and the third row.

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right|$$

is equivalent to

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right|$$

Linear equations

$$\left|\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right|$$

We see that the rank of A is 2. Hence it is smaller than the rank 3 of this system. Hence the system has no solution.
Indeed the system

$$\begin{array}{ccccccc} x_1 & & & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 1 \\ & & & & 0 & = & -1 \end{array}$$

contains a contradiction.

This week there will be an exercise class tomorrow!

Have a nice week!