Mathematics

IBS

Cornelia Busch

ETH Zürich

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Mappings and matrices

Let

$$\begin{aligned} f: \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax = y \end{aligned}$$

Then A is an $m \times n$-matrix.

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

Matrices

Let the linear function

$$\begin{array}{rccl} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f(x) := Ax\,. \end{array}$$

be defined by the 2×3-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

and let the linear function

$$\begin{array}{rccl} g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & x & \longmapsto & g(x) := Bx\,. \end{array}$$

be defined by the 3×2-matrix B.

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

Matrices

Then

$$\begin{aligned} f \circ g : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto f\big(g(x)\big) := ABx\,. \end{aligned}$$

and

$$\begin{aligned} g \circ f : \quad \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto g\big(f(x)\big) := BAx\,. \end{aligned}$$

Product of two matrices

Let A be a 2×3-matrix and B a 3×2-matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

Then $C = AB$ is a 2×2-matrix.

$$\begin{aligned} C &= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \end{aligned}$$

The component c_{ij} in $C = AB$ is given by the scalar product of the i-th row of A with the j-th column of B.

Product of two matrices

$$\begin{pmatrix} c_{11} & * \\ * & * \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ * & * & * \end{pmatrix} \begin{pmatrix} b_{11} & * \\ b_{21} & * \\ b_{31} & * \end{pmatrix}$$

$$\begin{pmatrix} * & c_{12} \\ * & * \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ * & * & * \end{pmatrix} \begin{pmatrix} * & b_{12} \\ * & b_{22} \\ * & b_{32} \end{pmatrix}$$

$$\begin{pmatrix} * & * \\ c_{21} & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & * \\ b_{21} & * \\ b_{31} & * \end{pmatrix}$$

$$\begin{pmatrix} * & * \\ * & c_{22} \end{pmatrix} = \begin{pmatrix} * & * & * \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} * & b_{12} \\ * & b_{22} \\ * & b_{32} \end{pmatrix}$$

Product of two matrices

The product of the $m \times l$-matrix

$$A = (a_{ik})_{\substack{i=1,\dots,m \\ k=1,\dots,l}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ a_{21} & a_{22} & \cdots & a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ml} \end{pmatrix}$$

and the $l \times n$-matrix

$$B = (b_{kj})_{\substack{k=1,\dots,l \\ j=1,\dots,n}} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{l1} & b_{l2} & \cdots & b_{ln} \end{pmatrix}$$

is the $m \times n$-matrix $AB = C$

Product of two matrices

$$C = (c_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \left(\sum_{k=1}^{l} a_{ik} b_{kj}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = AB\,.$$

The component c_{ij} in AB is given by the scalar product of the i-th row of A with the j-th column of B.

In general

$$A\,B \neq B\,A\,.$$

Sum and scalar multiplication

Given two $m \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij})$ we define the sum $A + B$ to be

$$A + B = (a_{ij} + b_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} .$$

Let $\alpha \in \mathbb{R}$ be a scalar then

$$\alpha A = (\alpha\, a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} .$$

The transpose of a matrix

We define the *transpose* A^t of the matrix

$$A := (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

to be the matrix

$$A^t := (a_{ji})_{\substack{j=1,\dots,n \\ i=1,\dots,m}}\,.$$

It follows immediately that

$$(A^t)^t = A\,.$$

The transpose of A is obtained in exchanging the rows with the columns. The i-th row of A is the i-th colum of A^t and the j-th column of A is the j-th row of A^t.

The transpose of a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad \Rightarrow \quad B^t = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

Matrices

For the system of two equations in two variables we consider a linear function

$$\begin{aligned} f: \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto f(x) := Ax\,. \end{aligned}$$

with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

a vector and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

a 2×2-matrix.

Matrices

The system

$$3x_1 + x_2 = 2$$
$$5x_1 + 2x_2 = 3$$

can be written

$$Ax = b$$

with

$$A := \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \quad b := \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The solution is

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Matrices

If f is invertible, then the solution of

$$f(x) = b$$

is

$$x = f^{-1}(b)\,.$$

Where the function f is the inverse of f.

- How can we see if f has an inverse?
- How can we find the inverse of f (if it exists)?

Invertible matrices

We consider

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f(x) := Ax\,. \end{array}$$

It would be convenient to have an inverse matrix A^{-1} to A such that

$$f^{-1}(y) = A^{-1}y$$

i.e.

$$\begin{aligned} x &= f^{-1}\big(f(x)\big) \\ &= A^{-1}Ax \end{aligned}$$

Indeed f is invertible if and only if A is invertible.

Invertible matrices

We first need the identity, i.e. the matrix that represents the function

$$\begin{array}{rccl} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & f(x) := x\,. \end{array}$$

For $n = 2$ this matrix is

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for $n = 3$ it is

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so on.

Invertible matrices

A given $n \times n$-matrix A is invertible if and only if there is a matrix A^{-1} such that

$$\mathbf{1} = A^{-1} A \,.$$

This is equivalent to the condition

$$\mathbf{1} = A A^{-1} \,.$$

Product of matrices

The product of matrices is not commutative.

In general

$$AB \neq BA$$

Invertible matrices

Can we see wether a matrix is invertible or not without computing the inverse?

Yes! This can be done with the determinant!

The determinant

The determinant $\det(A)$ of a $n \times n$-matrix A is a function that maps the matrix to an element of the underlying field ($\mathbb{R}$). It satisfies the following properties for $1 \leqslant i, j \leqslant n$

1. For $a \in \mathbb{R}$
$$\det(\alpha_1, \ldots, a\,\alpha_i, \ldots, \alpha_n) = a \cdot \det(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$$
2. With $\alpha_i = \alpha'{}_i + \alpha''{}_i$
$$\det(\alpha_1, \ldots, \alpha_i' + \alpha_i'', \ldots, \alpha_n) = \det(\alpha_1, \ldots, \alpha_i', \ldots, \alpha_n) + \det(\alpha_1, \ldots, \alpha_i'', \ldots, \alpha_n)\,.$$
3. Let $\alpha_i = \alpha_j$ for $i \neq j$. Then
$$\det(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_n) = 0\,.$$
4. Let $e_1 \ldots, e_n$ be the standard basis vectors, then
$$\det(e_1, \ldots, e_n) = 1\,.$$

The determinant: properties

The determinant of a $n \times n$-matrix

$$(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$$

is 0 if and only if the vectors $\alpha_1, \dots, \alpha_n$ are linearly dependent.
For the transpose of A we have

$$\det A = \det A^t \,.$$

For the product of the $n \times n$-matrices A and B

$$\det(AB) = \det A \cdot \det B \,.$$

Let $\alpha \in \mathbb{R}$ be a scalar, then

$$\det(\alpha A) = \alpha^n \det A \,.$$

The determinant: properties

The determinant of a $n \times n$-matrix

$$(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$$

is nonzero if and only if the vectors $\alpha_1, \ldots, \alpha_n$ are linearly independent.

In this case the square matrix is invertible.

The determinant: computation

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\,.$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}\,.$$

The determinant: computation

$$\det \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = 6 - 5 = 1\,.$$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 - 0 = 1\,.$$

$$\det \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = 2 - 2 = 0\,.$$

The determinant: computation

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11} \,. \end{array}$$

The determinant: computation

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}\,. \end{array}$$

The determinant: computation

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11} \, . \end{array}$$

The determinant: computation

We can also choose a column or a row and develop by this column or row.
We choose the first column. Then

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (-1)^{1+1} a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{2+1} a_{21} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{3+1} a_{31} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

The determinant: computation

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (-1)^{1+1} a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{2+1} a_{21} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{3+1} a_{31} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

The determinant: computation

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (-1)^{1+1} a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{2+1} a_{21} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{3+1} a_{31} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

The determinant: computation

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (-1)^{1+1} a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{2+1} a_{21} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{3+1} a_{31} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

The determinant: computation

This method can be generalised to $n \times n$-matrices.

A good strategy is to choose a column or a row with many 0.

The determinant: computation

$$
\begin{aligned}
\det \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} &= (-1)^{2+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + (-1)^{3+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \\
&= (-1+1) - (-1+2) \\
&= -1
\end{aligned}
$$

The determinant: invertible matrices

A square matrix

$$A \text{ is invertible} \quad \Leftrightarrow \quad \det A \neq 0$$

Inverse of a matrix

The inverse of a 2×2-matrix is computed as follows.
Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be invertible, i.e. $\det A = ad - bc \neq 0$.
Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a matrix

The inverse of a 2×2-matrix is computed as follows.
The matrix
$$A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$
is invertible, i.e. $\det A = 6 - 5 = 1$.
Then
$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

Inverse of a matrix

For a 3 $\times$ 3-matrix we have the following.
Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

If the determinant $\det A$ of A is nonzero, then the matrix has an inverse.

$$\begin{aligned} \det A = \; & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11} \,. \end{aligned}$$

Inverse of a matrix

We now consider the transpose A^t of A.

$$A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

The entry at the position ij in the inverse A^{-1} is

$$(-1)^{i+j} \frac{1}{\det A} \det M_{ij}$$

where M_{ij} is the ij-minor in A^t, that is the 2×2-matrix that is left when we take off the ith row and the jth column.

Inverse of a matrix

$$A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} (-1)^{1+1} \det \begin{pmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{pmatrix} & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

Inverse of a matrix

$$A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} (-1)^{1+1} \det \begin{pmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{pmatrix} & * & * \\ (-1)^{2+1} \det \begin{pmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{pmatrix} & * & * \\ * & * & * \end{pmatrix}$$

Inverse of a matrix

$$A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} (-1)^{1+1} \det \begin{pmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{pmatrix} & * & * \\ (-1)^{2+1} \det \begin{pmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{pmatrix} & * & * \\ * & (-1)^{3+2} \det \begin{pmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{pmatrix} & * \end{pmatrix}$$

Inverse of a matrix

$$A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \det \begin{pmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{pmatrix} & -\det \begin{pmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{pmatrix} & \det \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \\ -\det \begin{pmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{pmatrix} & \det \begin{pmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{pmatrix} & -\det \begin{pmatrix} a_{11} & a_{21} \\ a_{13} & a_{23} \end{pmatrix} \\ \det \begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix} & -\det \begin{pmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{pmatrix} & \det \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \end{pmatrix}$$

The inverse: computation

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

We already computed $\det A = -1$. Now

$$A^t = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} &= (-1)^{2+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + (-1)^{3+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \\ &= (-1+1) - (-1+2) \\ &= -1 \end{aligned}$$

The inverse: computation

$$A^t = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$A^{-1} = (-1) \cdot \begin{pmatrix} \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & \det\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \\ -\det\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \\ \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

The inverse: computation

$$A^{-1} = (-1) \cdot \begin{pmatrix} \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & \det\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \\ -\det\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \\ \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$A^{-1} = (-1) \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$