Mathematics

IBS

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Basis

A maximal set of linearly independent vectors $\tau_1, \ldots, \tau_n$ in a vector space is called a basis.

The standard-basis of $\mathbb{R}^n$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Basis

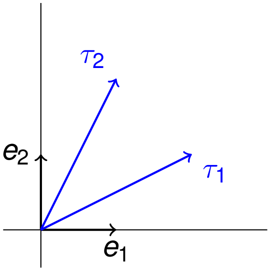
The standard basis of $\mathbb{R}^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The vectors

$$\tau_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \tau_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

are linearly independent since they are not parallel. They form a basis of $\mathbb{R}^2$.



Basis

For

$$\tau_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \tau_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

we check that

$$\begin{aligned} e_1 &= \frac{2}{3}\tau_1 - \frac{1}{3}\tau_2 \\ e_2 &= -\frac{1}{3}\tau_1 + \frac{2}{3}\tau_2 \end{aligned}$$

and this shows that τ_1, τ_2 form a basis since e_1 and e_2 form a basis.

Basis

The vectors τ_1 and τ_2 are represented in the basis e_1, e_2:

$$\tau_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2e_1 + e_2 \quad \text{and} \quad \tau_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e_1 + 2e_2$$

In the basis τ_1, τ_2 they are written

$$\tau_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; \tau_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Basis

The previous slide shows that the coordinates of a vector $x \in \mathbb{R}^n$ depend on the basis of $\mathbb{R}^n$.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Change of basis

We define the mapping

$$\begin{array}{rrcl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f(x) \end{array}$$

by

$$f(e_1) := 2e_1 + 7e_2\,, \quad f(e_2) := e_1 + 4e_2\,.$$

This defines the mapping because f is linear, i.e., $f(x_1 e_1 + x_2 e_2) = x_1 f(e_1) + x_2 f(e_2)$.

Change of basis

In the basis e_1, e_2 we can write f as $x \longmapsto f(x) := Ax$ with

$$A = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

since

$$e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 2 \\ 7 \end{pmatrix} = 2e_1 + 7e_2$$

$$e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 4 \end{pmatrix} = e_1 + 4e_2$$

Change of basis

We now want to express the mapping f in the basis $\tau_1 := 2e_1 + e_2$ and $\tau_2 := e_1 + 2e_2$. Using their representation in the standard basis, we determine the image of τ_1 and τ_2.

$$f(\tau_1) = A\tau_1 = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 18 \end{pmatrix} = 5e_1 + 18e_2$$
$$f(\tau_2) = A\tau_2 = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \end{pmatrix} = 4e_1 + 15e_2$$

Change of basis

The rows in the matrix B that describes f in the basis τ_1, τ_2, are the images $f(\tau_1)$, $f(\tau_2)$ of these vectors written in the basis τ_1, τ_2.

Hence we determine $b_{11}, b_{21}, b_{12}, b_{22} \in \mathbb{R}$ such that

$$f(\tau_1) = b_{11}\tau_1 + b_{21}\tau_2$$
$$f(\tau_2) = b_{12}\tau_1 + b_{22}\tau_2$$

Then

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Change of basis

We therefore solve

$$f(\tau_1) = b_{11}\tau_1 + b_{21}\tau_2$$
$$f(\tau_2) = b_{12}\tau_1 + b_{22}\tau_2$$

that is the system

$$\begin{pmatrix} 5 \\ 18 \end{pmatrix} = b_{11}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b_{21}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 4 \\ 15 \end{pmatrix} = b_{12}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b_{22}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Change of basis

We get

$$f(\tau_1) = -\frac{8}{3}\tau_1 + \frac{31}{3}\tau_2$$
$$f(\tau_2) = -\frac{7}{3}\tau_1 + \frac{26}{3}\tau_2$$

Hence the matrix B is

$$B = \begin{pmatrix} -\frac{8}{3} & -\frac{7}{3} \\ \frac{31}{3} & \frac{26}{3} \end{pmatrix}$$

Change of basis

There is a fast way to find B when we have A and the new basis.

We define the transformation matrix T that maps the old basis to the new basis τ_1, τ_2.
This matrix T is written in the old basis.
Let

$$\tau_1 = \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix}.$$

Then

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

Change of basis

The matrix B that describes the mapping f in the basis $\tau_1, \ldots, \tau_n$ is given by

$$B = T^{-1} A T$$

and herewith

$$T B T^{-1} = A .$$

Here we use that in general the product of $n \times n$-matrices is not commutative, i.e., $MN \neq NM$.

Change of basis

In the example the matrix T is

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and its inverse is

$$T^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Change of basis

Hence

$$AT = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 18 & 15 \end{pmatrix}$$

and

$$TB = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{8}{3} & -\frac{7}{3} \\ \frac{31}{3} & \frac{26}{3} \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 18 & 15 \end{pmatrix}$$

You might check that $B = T^{-1}AT$.

Change of basis: An example

In the standard basis e_1, e_2, e_3 the mapping

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

is given by

$$x \longmapsto f(x) := Ax$$

with

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 4 \end{pmatrix}, \quad x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We want to write f in a new basis.

Change of basis: An example

We choose a new basis

$$\tau_1 := e_1 - e_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \tau_2 := e_2 - e_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tau_3 := e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then the transformation from the standard basis to the basis $\{\tau_1, \tau_2, \tau_3\}$ is given by the matrix

$$T := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

We have $\det T = 1$.

Change of basis: An example

In the basis $\{\tau_1, \tau_2, \tau_3\}$ the mapping f is given by

$$x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto T^{-1} A T\, x\,.$$

Here

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 4 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Change of basis: An example

Then

$$
\begin{aligned}
T^{-1}AT &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} .
\end{aligned}
$$

This week there will be an exercise class on Friday!

See you tomorrow!