Mathematics

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Linear algebra and geometry

How can we see what a given mapping

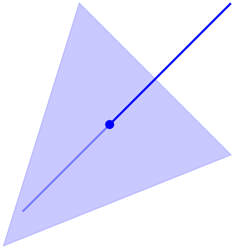
$$\begin{aligned} f: \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto f(x) := Ax \end{aligned}$$

does?

It may be a projection, a rotation, a reflection, a scaling, ...

Linear algebra and geometry

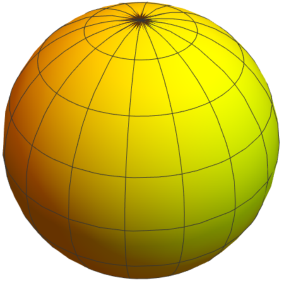
An interesting case is the one where a subspace is mapped onto itself.



The rotation about the blue axis keeps any point of the axis fixed and maps any point in the blue plane to a point in the blue plane. Hence the plane and the axis are subspaces of $\mathbb{R}^3$ that are mapped onto themselves by the rotation.

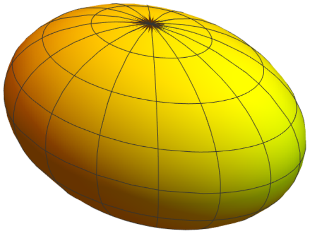
Scalings

By a scaling a sphere . . .



Scalings

...may become an ellipsoid.



Directions and scalars

For the simplicity of notation we choose $n = 3$.

We know that the columns of the matrix A that defines the mapping

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & x & \longmapsto & f(x) := Ax \end{array}$$

are the images $f(e_1), f(e_2), f(e_3)$ of the elements of the basis e_1, e_2, e_3 written in this basis.

Hence the matrix A depends on the choice of the basis.

Directions and scalars

Let

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & x & \longmapsto & f(x) \end{array}$$

be linear and let $v_1, v_2, v_3 \in \mathbb{R}^3$ be linearly independent and such that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ exist with

$$\begin{array}{ccc} v_1 & \longmapsto & f(v_1) = \lambda_1 v_1 \\ v_2 & \longmapsto & f(v_2) = \lambda_2 v_2 \\ v_3 & \longmapsto & f(v_3) = \lambda_3 v_3 \end{array}$$

Directions and scalars

Then

$$f(v_1) = \lambda_1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$$
$$f(v_2) = 0 \cdot v_1 + \lambda_2 \cdot v_2 + 0 \cdot v_3$$
$$f(v_3) = 0 \cdot v_1 + 0 \cdot v_2 + \lambda_3 \cdot v_3$$

hence in the basis v_1, v_2, v_3

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} = f(v_1)$$
$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix} = f(v_2)$$
$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix} = f(v_3)$$

Directions and scalars

If such vectors v_1, v_2, v_3 exist, they satisfy

$$\begin{aligned} f(v_1) &= \lambda_1 v_1 \\ f(v_2) &= \lambda_2 v_2 \\ f(v_3) &= \lambda_3 v_3 \end{aligned}$$

Hence

$$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ Av_3 &= \lambda_3 v_3 \end{aligned}$$

Directions and scalars

The matrix B that defines the mapping

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & x & \longmapsto & f(x) := Bx \end{array}$$

in the basis v_1, v_2, v_3 is

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

This is a diagonal matrix.

Directions and scalars

For a given mapping that is defined by

$$\begin{array}{rccl} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & x & \longmapsto & f(x) := Ax \end{array}$$

in the standard basis e_1, e_2, e_3, we want to find a new basis v_1, v_2, v_3 such that in this new basis the mapping f is given by $f(x) = Bx$ with

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

Eigenvalues and eigenvectors

The coefficients

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$$

are called the eigenvalues of *A*.

The vectors

$$v_1, v_2, v_3 \in \mathbb{R}^3$$

are called the eigenvectors of *A*.

Eigenvalues and eigenvectors

We now determine the eigenvalues and eigenvectors of f or equivalently of A.
We solve

$$Av = \lambda v$$

that is equivalent to

$$Av - \lambda v = (A - \lambda \mathbf{1})v = 0$$

There exist nonzero solutions v of $(A - \lambda \mathbf{1})v = 0$ if and only if $(A - \lambda \mathbf{1})$ is not invertible, i.e., if and only if

$$\det(A - \lambda \mathbf{1}) = 0\,.$$

Eigenvalues and eigenvectors

Hence we find scalars λ such that

$$p_A(\lambda) = \det(A - \lambda \mathbf{1}) = 0\,.$$

The determinant is a polynomial of degree 3 (resp. n) in λ. It is called the characteristic polynomial of A.

For each solution λ we determine $v \neq 0$ that satisfies

$$(A - \lambda \mathbf{1})v = 0\,.$$

The solution v is not unique. It belongs to a subspace of $\mathbb{R}^3$ (resp. $\mathbb{R}^n$).

Eigenvalues and eigenvectors

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be the zeros[1] of the characteristic polynomial

$$p_A(\lambda) = \det(A - \lambda \mathbf{1}) .$$

If the zeros of $p_A(\lambda)$ are pairwise different, then we can find a basis in which the matrix corresponding to f is diagonal.
This is not always true if the zeros are not pairwise different. As an example we consider

$$p_A(\lambda) = (\lambda - \lambda_1)^2 \cdot (\lambda - \lambda_2)$$

with $\lambda_1 \neq \lambda_2$. Then a diagonal matrix exists if and only if we can find two linearly independent eigenvectors to the eigenvalue λ_1. There are some cases where all the eigenvectors to the eigenvalue λ_1 are linearly dependent.

[1] We do not consider the case of complex eigenvalues.

Eigenvalues and eigenvectors

We consider $\mathbb{R}^n$ with the standard basis $e_1, \ldots, e_n$. Let

$$\begin{aligned} f: \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto f(x) := Ax \end{aligned}$$

be a linear mapping. Then the columns in A are the images $f(e_1), \ldots, f(e_n)$ of the standard basis.
In the basis of eigenvectors $v_1, \ldots, v_n$ the mapping f is given by

$$x \longmapsto f(x) = Bx$$

The change of basis allows us to transform A in order to get the diagonal matrix B.

Change to the basis of eigenvectors

The transformation matrix T for the change of basis to the basis of eigenvectors contains the eigenvectors of A in the columns.

$$T = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}$$

Computation of eigenvalues and eigenvectors

We compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

Computation of eigenvalues and eigenvectors

Step I: Determine the characteristic polynomial.

The characteristic polynomial $p_A(\lambda)$ is

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda \mathbf{1}) \\ &= \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 1 \\ &= \lambda^2 - 4\lambda + 2 \end{aligned}$$

Computation of eigenvalues and eigenvectors

Step II: Compute the eigenvalues.

The zeros of the characteristic polynomial $p_A(\lambda) = \lambda^2 - 4\lambda + 2$ are

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$$

$$\lambda_1 = 2 + \sqrt{2}\,, \quad \lambda_2 = 2 - \sqrt{2}\,.$$

Computation of eigenvalues and eigenvectors

Step III: Compute the eigenvectors.

We first determine the eigenvector v_1 to the eigenvalue $\lambda_1 = 2 + \sqrt{2}$. We solve $Av_1 = \lambda_1 v_1$ that is equivalent to $(A - \lambda_1 \mathbf{1})v_1 = 0$. We compute

$$A - \lambda_1 \mathbf{1} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2+\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix}$$

Computation of eigenvalues and eigenvectors

For $v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$(A - \lambda_1 \mathbf{1})v_1 = \begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-1-\sqrt{2})x + y \\ x + (1-\sqrt{2})y \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

hence

$$\begin{aligned} (-1-\sqrt{2})x + y &= 0 \\ x + (1-\sqrt{2})y &= 0 \end{aligned}$$

and, since $(-1-\sqrt{2})(1-\sqrt{2}) = 1$, this system is equivalent to

$$y = (1+\sqrt{2})x$$

and we choose

$$v_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} .$$

Computation of eigenvalues and eigenvectors

We now determine the eigenvector v_2 to the eigenvalue $\lambda_2 = 2 - \sqrt{2}$. We solve $(A - \lambda_2 \mathbf{1})v_2 = 0$. We compute

$$\begin{aligned} A - \lambda_2 \mathbf{1} &= \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2-\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix} \end{aligned}$$

Computation of eigenvalues and eigenvectors

For $v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$(A - \lambda_2 \mathbf{1}) v_2 = \begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} (-1+\sqrt{2})x + y \\ x + (1+\sqrt{2})y \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and, since $(-1+\sqrt{2})(1+\sqrt{2}) = 1$, this system is equivalent to

$$y = (1-\sqrt{2})x$$

and we choose

$$v_2 = \begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix}$$

Computation of eigenvalues and eigenvectors

Let

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ x & \longmapsto & Ax \end{array}$$

be the mapping that is defined by A in the standard basis. Then, in the basis of eigenvectors v_1, v_2, the mapping f is given by

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{pmatrix}.$$

Computation of eigenvalues and eigenvectors

The transformation matrix is

$$T = \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix}$$

and its inverse is

$$T^{-1} = \frac{1}{-4-2\sqrt{2}} \begin{pmatrix} -1 & -1-\sqrt{2} \\ -1-\sqrt{2} & 1 \end{pmatrix} .$$

It is now easy to check that

$$B = T^{-1} A T .$$

An example in $\mathbb{R}^3$

Consider the matrix

$$A := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}.$$

- Determine the eigenvalues of A. Compute the corresponding eigenvectors.
- Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping that is defined in the standard basis of $\mathbb{R}^3$ by $f(x) = Ax$. Determine the matrix B that represents f in the basis of eigenvectors.

An example in $\mathbb{R}^3$

The characteristic polynomial is

$$\det(A - \lambda\mathbf{1}) = \det\begin{pmatrix} -1-\lambda & 0 & 0 \\ 0 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{pmatrix}$$

$$= (-1-\lambda)\cdot\det\begin{pmatrix} -\lambda & 3 \\ 3 & -\lambda \end{pmatrix}$$

$$= (-1-\lambda)\cdot(\lambda^2 - 9) = (1+\lambda)\cdot(9-\lambda^2)$$

$$= (1+\lambda)(3-\lambda)(3+\lambda)$$

The eigenvalues are

$$\lambda_1 = -1\,, \quad \lambda_2 = 3\,, \quad \lambda_3 = -3\,.$$

An example in $\mathbb{R}^3$

An eigenvector v_{-1} to the eigenvalue $\lambda_1 = -1$ is a solution of

$$(A - (-1) \cdot \mathbf{1}) \cdot v_{-1} = 0$$

We have

$$A + \mathbf{1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

and

$$(A + \mathbf{1})v_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \Rightarrow \quad v_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

An example in $\mathbb{R}^3$

An eigenvector v_3 to the eigenvalue $\lambda_2 = 3$ is a solution of

$$(A - 3 \cdot \mathbf{1}) \cdot v_3 = 0$$

We have

$$A - 3 \cdot \mathbf{1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$$

and

$$(A - 3 \cdot \mathbf{1}) v_3 = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \Rightarrow \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

An example in $\mathbb{R}^3$

An eigenvector v_{-3} to the eigenvalue $\lambda_3 = -3$ is a solution of

$$(A + 3 \cdot \mathbf{1}) \cdot v_{-3} = 0$$

We have

$$A + 3 \cdot \mathbf{1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}$$

and

$$(A + 3 \cdot \mathbf{1}) v_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \Rightarrow \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

An example in $\mathbb{R}^3$

The corresponding eigenvectors are

$$v_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_{-3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

The transformation matrix T is

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

An example in $\mathbb{R}^3$

The function f is given in the basis $\{v_{-1}, v_3, v_{-3}\}$ by

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Another example in $\mathbb{R}^3$

We compute the eigenvalues and the eigenvectors of the matrix

$$A := \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{pmatrix}.$$

Another example in $\mathbb{R}^3$

The characteristic polynomial of A is

$$\det(A - \lambda \mathbf{1}) = \det \begin{pmatrix} -\frac{1}{3} - \lambda & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} - \lambda & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} - \lambda \end{pmatrix}$$

$$= -\left(\frac{1}{3} + \lambda\right)^3 + 2 \cdot \left(\frac{2}{3}\right)^3 + 3 \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3} + \lambda\right)$$

$$= -\left(\frac{1}{3^3} + \frac{1}{3}\lambda + \lambda^2 + \lambda^3\right) + \frac{2^4}{3^3} + \frac{2^2}{3^2} + \frac{2^2}{3}\lambda$$

$$= -\lambda^3 - \lambda^2 + \lambda + 1$$

Another example in $\mathbb{R}^3$

The eigenvalues satisfy

$$-\lambda^3 - \lambda^2 + \lambda + 1 = 0$$

We see that $\lambda_1 = 1$ and division of polynomials shows that

$$-\lambda^3 - \lambda^2 + \lambda + 1 = (1 - \lambda)(\lambda^2 + 2\lambda + 1)$$

Now we solve $\lambda^2 + 2\lambda + 1 = 0$ and get

$$-\lambda^3 - \lambda^2 + \lambda + 1 = (1 - \lambda)(1 + \lambda)^2$$

Hence the eigenvalues are

$$\lambda_1 = 1\,, \quad \lambda_2 = \lambda_3 = -1$$

Another example in $\mathbb{R}^3$

Since A doesn't have three different eigenvalues, we do not know if the matrix A is diagonalisable.

- If we find two linearly independent eigenvectors to the eigenvalue -1, then A is diagonalisable.
- If all the eigenvectors to -1 are linearly dependent, then A is not diagonalisable.

Another example in $\mathbb{R}^3$

We first determine an eigenvector to the eigenvalue $\lambda_1 = 1$.

$$(A - \mathbf{1})v_1 = \frac{1}{3}\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Has the solution

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Another example in $\mathbb{R}^3$

For the eigenvector to the eigenvalue $\lambda_1 = -1$ we solve

$$(A + \mathbf{1})v = \frac{1}{3}\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

This is equivalent to

$$x + y + z = 0$$

As we have one equation for three variables we can find two linearly independent solutions.

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Another example in $\mathbb{R}^3$

The transformation matrix T is

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

Its inverse is

$$T^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

Check that

$$T^{-1}AT = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This week there will be an exercise class on Friday!

Have a nice week!