Mathematics

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Applications of linear algebra

Linear algebra helps to find the local extrema of functions in more than one variable.

Local extrema

Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a function. A point $x \in U$ is called *local maximum* of f if an environment $V \subset U$ of x exists with

$$f(x) \geqslant f(y) \quad \text{for all } y \in V\,.$$

A point $x \in U$ is called *local minimum* of f if an environment $V \subset U$ of x exists with

$$f(x) \leqslant f(y) \quad \text{for all } y \in V\,.$$

The Hessian

Let $U \subseteq \mathbb{R}^n$ be an open set and let

$$f : \begin{array}{rcl} U & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & f(x_1, \ldots, x_n) \end{array}$$

be a function whose first and second partial derivatives exist and are continuous.

The *Hessian matrix* of f in $x \in U$ is the $n \times n$-matrix

$$(\operatorname{Hess} f)(x) := \left(\frac{\partial^2}{\partial x_i \partial x_j} f(x) \right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} = \big(f_{ij}(x)\big)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} .$$

This matrix is symmetric since for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$

$$f_{ij}(x) = f_{ji}(x) .$$

Symmetric matrices

An $n \times n$-matrix A is called symmetric if and only if $A = A^t$.

A symmetric matrix A is

- *positive definite* if all its eigenvalues are positive,
- *negative definite* if all its eigenvalues are negative,
- *indefinite* if there is at least one positive and one negative eigenvalue.

Local extrema

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a partial differentiable function. If f has a local extremum in the point x (i.e. a local maximum or a local minimum), then

$$\nabla f(x) = 0\,.$$

As for functions in one variable, the reverse is not true.

Local extrema

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ a function whose first and second partial derivatives exist and are continuous. Let $x \in U$ with

$$\nabla f(x) = 0\,.$$

- If $(\mathrm{Hess}\, f)(x)$ is positive definite, then f has a local minimum in x.
- If $(\mathrm{Hess}\, f)(x)$ is negative definite, then f has a local maximum in x.
- If $(\mathrm{Hess}\, f)(x)$ is indefinite, then f doesn't have a local extremum in x.

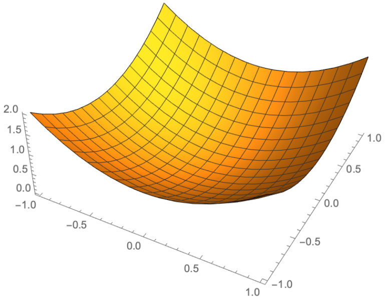
In the other cases there may not be a local extremum.

Examples

The function $f(x, y) := x^2 + y^2$ has a local minimum in $(0, 0)$ since $\nabla f(0, 0) = (0, 0)$ and the Hessian

$$(\text{Hess}\, f)(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive definite.

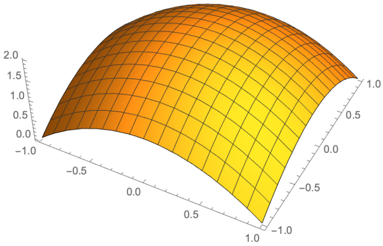


Examples

The function $g(x, y) := 2 - x^2 - y^2$ has a local maximum in $(0, 0)$ since $\nabla g(0, 0) = (0, 0)$ and the Hessian

$$(\text{Hess}\, g)(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

is negative definite.

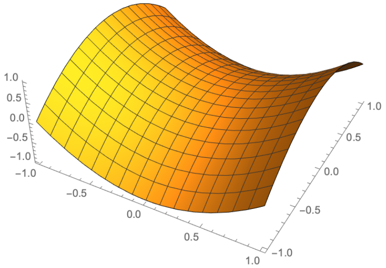


Examples

The function $h(x, y) := x^2 - y^2$ satisfies $\nabla h(0,0) = (0,0)$ and the Hessian

$$(\operatorname{Hess} h)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

is indefinite. The function has a saddle point in $(0,0)$.



Examples

For the function $f(x, y) := x^3 + y^3$ we have $\nabla f(x, y) = (3x^2, 3y^2)$ and the Hessian

$$(\text{Hess}\, f)(x, y) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$$

In $(0, 0)$ we have $\nabla f(0, 0) = (0, 0)$ and the Hessian

$$(\text{Hess}\, f)(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Problem

Consider the function

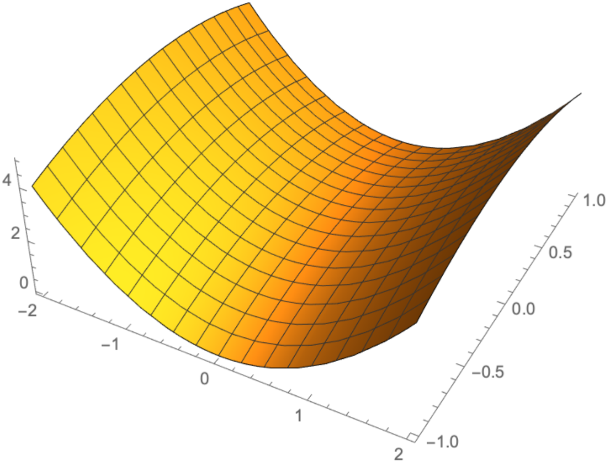
$$f(x, y) = 1 + x^2 - y^2$$

on the set

$$S = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{4} + y^2 \leqslant 1 \right\}.$$

Find the values of (x, y) at which the function f attains its global extrema.

Problem



Problem

We first consider the inner points of the ellipse. The gradient of $f(x,y) = 1 + x^2 - y^2$ is

$$\nabla f(x,y) = (2x, -2y)\,.$$

The gradient is zero if and only if $(x,y) = (0,0)$. Hence the origin is the only candidate for an extremum in the inner of the ellipse.

The Hessian in (x,y) is

$$(\mathrm{Hess}\, f)(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\,.$$

In this case the Hessian is constant.

Problem

We know that

$$\nabla f(0,0) = (2 \cdot 0, -2 \cdot 0) = (0,0)\,.$$

The Hessian in $(0,0)$ is

$$(\mathrm{Hess}\, f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

and the point $(0,0)$ is a saddle point of f.

Problem

We now consider f on the boundary $\frac{x^2}{4} + y^2 = 1$ and substitute

$$f(x, y) = x^2 + \underbrace{1 - y^2}_{=\frac{x^2}{4}} = \frac{5}{4}\,x^2\,.$$

The first and second derivatives of the function

$$g(x) = \frac{5}{4}\,x^2$$

are

$$g'(x) = \frac{5}{2}\,x \quad \text{and} \quad g''(x) = \frac{5}{2}\,.$$

We consider the zeros of $g'(x)$.

Problem

We consider the values of x with

$$g'(x) = \frac{5}{2}x = 0 \quad \Longleftrightarrow \quad x = 0\,.$$

Since $g''(0) = \frac{5}{2} > 0$ these are local minima with

$$f(0,-1) = 0 = g(0) \quad \text{and} \quad f(0,1) = 0 = g(0)\,.$$

Since g is a function on the interval $[-2,2]$, we have to study the function g on the boundary of this interval.

Problem

We study $g(x) = \frac{5}{4}x^2$ in $x = -2$ and $x = 2$.

$$g(-2) = 5 = f(-2, 0)$$

and

$$g(2) = 5 = f(2, 0)$$

These are local maxima.

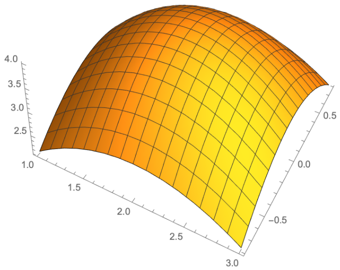
The global minima are $(0, -1)$ and $(0, 1)$ and the global maxima are $(-2, 0)$ and $(2, 0)$.

Another problem

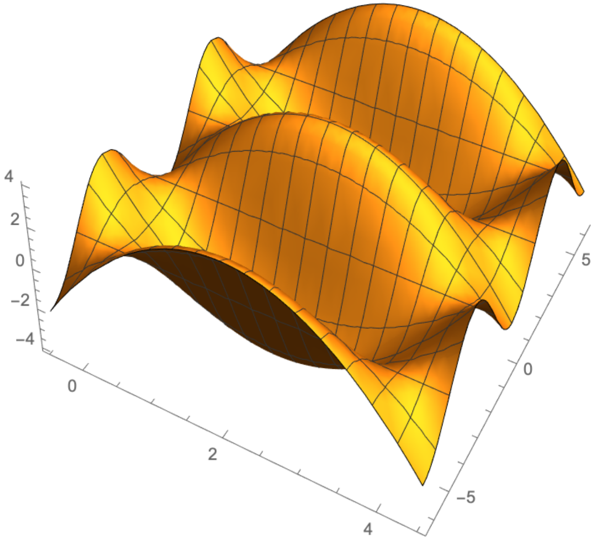
Find the absolute extrema of the surface

$$f(x,y) = (4x - x^2)\cos(y)$$

on the rectangular plate $1 \leqslant x \leqslant 3$, $-\frac{\pi}{4} \leqslant y \leqslant \frac{\pi}{4}$.



Another problem



Another problem

The gradient of the function $f(x, y) = (4x - x^2)\cos(y)$ is

$$\nabla f(x, y) = \big((4 - 2x)\cos(y),\ -(4x - x^2)\sin(y)\big)$$

On our plate $1 \leqslant x \leqslant 3$, $-\frac{\pi}{4} \leqslant y \leqslant \frac{\pi}{4}$

$$f_x(x, y) = 0 \quad \text{only for} \quad x = 2$$
$$f_y(x, y) = 0 \quad \text{only for} \quad y = 0$$

We have

$$f(2, 0) = 4\,.$$

Another problem

$$\nabla f(x, y) = \big((4 - 2x)\cos(y),\ -(4x - x^2)\sin(y)\big)$$

$$(\operatorname{Hess} f)(x, y) = \begin{pmatrix} -2\cos(y) & (2x - 4)\sin(y) \\ (2x - 4)\sin(y) & -(4x - x^2)\cos(y) \end{pmatrix}$$

$$(\operatorname{Hess} f)(2, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

Another problem

We need to check the boundary.

$$f(1,y) = 3\cos(y)\,, \quad \text{with } -\frac{\pi}{4} \leqslant y \leqslant \frac{\pi}{4}$$

$$f(3,y) = 3\cos(y)\,, \quad \text{with } -\frac{\pi}{4} \leqslant y \leqslant \frac{\pi}{4}$$

$$f\left(x, -\frac{\pi}{4}\right) = \frac{4x - x^2}{\sqrt{2}}\,, \quad \text{with } 1 \leqslant x \leqslant 3$$

$$f\left(x, \frac{\pi}{4}\right) = \frac{4x - x^2}{\sqrt{2}}\,, \quad \text{with } 1 \leqslant x \leqslant 3$$

Another problem

The function $3\cos(y)$ only has a maximum at $y = 0$ and

$$f(1,0) = f(3,0) = 3\,.$$

Next the function $\frac{4x-x^2}{\sqrt{2}}$ only has a maximum at $x = 2$ and

$$f\left(2,-\frac{\pi}{4}\right) = f\left(2,\frac{\pi}{4}\right) = \frac{4}{\sqrt{2}}\,.$$

Another problem

Finally

$$f\left(1,-\frac{\pi}{4}\right)=f\left(1,\frac{\pi}{4}\right)=f\left(3,-\frac{\pi}{4}\right)=f\left(3,\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$$

- The absolute maximum is 4 at $(2,0)$.
- the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(1,-\frac{\pi}{4}\right)$, $\left(1,\frac{\pi}{4}\right)$, $\left(3,-\frac{\pi}{4}\right)$ and $\left(3,\frac{\pi}{4}\right)$.

That’s all Folks!