1.1. Classification of PDEs Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\cos(x+y)u_x - \sin(y)u_{yy} = 0$.

(b) $\Delta(u - u_y) = 4$.

(c) $(u_{xx} + 1)^3 = x^3 + 2$.

(d) $u_{xx}u_{yy} - (u_{xy})^2 = 1$.

(e) $u_{xy} - u_x u_y = 2$.

1.2. Solutions to ODEs Solve the following ODEs.

(a) $x'(t) + \lambda x(t) = 0$, with $x(0) = x_0$.

(b) $x'(t) + \lambda x(t) = 1$, with $x(0) = x_0$.

(c) $x'(t) + x(t) = t$, with $x(0) = 1$.

(d) $x'(t) + x(t) = e^t$, with $x(0) = 1$.

(e) $x''(t) + \lambda^2 x(t) = 0$, find a general solution.

1.3. Nonexistence of solutions Show that there is not a smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{cases} u_x = xy\,, \\ u_y = x^2\,. \end{cases}$$

1.4. Existence of infinite solutions Consider the Cauchy problem (PDE + imposed initial data)

$$\begin{cases} u_x + u_y = 0, & (x,y) \in \mathbb{R}^2, \\ u(x,x) = 1, & x \in \mathbb{R}. \end{cases}$$

(a) Show that there exists an infinite amount of smooth functions $u : \mathbb{R}^2 \to \mathbb{R}$ solving it.

(b) Find a solution u such that for all $(x,y) \in \mathbb{R}^2$, $u(x,y) = u(y,x)$, $u(x,y) > 0$ and $\lim_{x\to+\infty} u(x,0) = 0$.

(c) Show that there is no alternating solution, i.e. satisfying $u(x,y) = -u(y,x)$ for all $(x,y) \in \mathbb{R}^2$.

1.5. Multiple Choice Determine correct answer(s) to each point.

(a) Let $A : \mathbb{R}^4 \to \mathbb{R}$ be a nontrivial ($A \not\equiv 0$) linear map. Then, the PDE $A(u, u_x, u_y, u_{xx}) = 0$ is *always*

- ◯ linear
- ◯ inhomogeneous
- ◯ homogeneous
- ◯ of second order
- ◯ of first order
- ◯ quasilinear

(b) The reaction-diffusion equation (widely used in biology, geology, chemistry, physics and ecology as a model for pattern formation) takes this form in its simple formulation: $u_t = \Delta u + R(u)$, where $R : \mathbb{R} \to \mathbb{R}$ accounts for local reactions, and $u = u(t, x, y)$ represents the unknown (density of a chemical substance/population etc) at time $t \geq 0$ and position $(x, y) \in \mathbb{R}^2$. This PDE is

- ◯ of third order
- ◯ of second order
- ◯ linear if R is an affine map
- ◯ inhomogeneous if R is a linear map
- ◯ fully nonlinear
- ◯ of first order

(c) Consider the PDE $\Delta u + \nabla u \cdot \nabla u = 0$, where with the "dot" we denote the usual scalar product in $\mathbb{R}^n$. Then, the function $v := e^u$ solves a PDE that is

- ◯ linear and homogeneous
- ◯ linear and inhomogeneous
- ◯ quasilinear
- ◯ fully nonlinear

(d) For a smooth vector field $F = (F^1, F^2, \ldots, F^n) : \mathbb{R}^n \to \mathbb{R}^n$, we denote with $\mathrm{div}(F)$ the divergence of F, given by $\mathrm{div}(F) := \sum_{i=1}^n F^i_{x_i}$. The following PDE

$$\mathrm{div}(\nabla(u^2)) = u,$$

is

- ◯ of third order and fully nonlinear
- ◯ linear
- ◯ quasilinear
- ◯ of second order

Extra exercises

1.6. Classification of PDEs, II Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\Delta(\Delta u) = 5u$.

(b) $10^{20}u + \sin(u_x) = u_{xx}$.

(c) $e^{\Delta u} = u$.

(d) $\partial_x(uu_y) = \partial_y(uu_x)$.

1.7. Solutions to PDEs Check whether each of the following PDEs has a solution u that is a polynomial and, if it exists, determine a polynomial that solves the PDE.

(a) $\Delta u = x + y$.

(b) $u_{xx} = -u$, with $u(0) = 1$.

(c) $u_{xx} + u_{xy} = \sin(x)$.

(d) $u_{xyx}^2 + u_{yxy} = e^u$.

(e) $u_{xx} + u_y + u_{xy} = x^2y$.