3.1. Characteristic method and initial conditions Consider the equation

$$xu_y - yu_x = 0.$$

For each of the following initial conditions, solve the problem in $y \geq 0$ whenever it is possible. If it is not, explain why.

(a) $u(x, 0) = x^2$.

(b) $u(x, 0) = x$.

(c) $u(x, 0) = x$ for $x > 0$.

3.2. Method of characteristic, local and global existence Consider the quasilinear, first order PDE

$$\begin{cases} u_x + \ln(u)u_y = u, & (x, y) \in \mathbb{R}^2, \\ u(x, 0) = e^x, & x \in \mathbb{R}, \end{cases}$$

(here $\ln(\cdot)$ stands for the natural logarithm).

(a) Check the transversality condition.

(b) Find an explicit solution, and check if the result matches the existence condition found in the previous point.

3.3. Multiple choice Cross the correct answer(s).

(a) Consider the first order linear PDE: $(x + e^y)u_x + u_y = x$. Then, the transversality condition is everywhere satisfied if

- ○ $u(0, y) = y$
- ○ $u(x, 0) = \sin(x)$
- ○ $u(x, x) = xy$
- ○ $u(x^2, x) = 0$

(b) Consider the first order quasilinear PDE: $xu_x + e^u u_y = 0$. Then, the transversality condition is satisfied if

- ○ $u(x, x^2) = \ln(1 + x^2)$, $x > 1$
- ○ $u(x, x^2) = \ln(1 + x^2)$, $x \geq 0$
- ○ $u(0, y) = y$
- ○ $u(x, 0) = h(x)$ for any function h

(c) For which values of $r > 0$ there exists a local solution for

$$xu_x + (u + y)u_y = x^3 + 2,$$

in a neighbourhood of the circle $C_r := \{\sqrt{x^2 + y^2} = r^2\}$, so that $u|_{C_r} \equiv -1$?

- ◯ $r > 1$
- ◯ $0 < r < 1$
- ◯ $r \geq 1$
- ◯ $r = 1$

(d) For which values of $a > 0$ there exists a local solution of

$$uu_x + (y + a)u_y = 2022,$$

in a neighbourhood of the ellipse $E_a := \{\frac{x^2}{a^2} + y^2 = 1\}$, so that $u|_{E_a} = x$?

- ◯ $a = 1$
- ◯ $0 < a < 1$
- ◯ $a > 0$
- ◯ $a \geq 1$

Extra exercises

3.4. Characteristic method and transversality condition Consider the transport equation

$$yu_x + uu_y = x.$$

(a) Solve the problem with initial condition $u(s, s) = -2s$, for $s \in \mathbb{R}$. For what domain of s does the transversality condition hold?

(b) Check the transversality condition with the initial value $u(s, s) = s$. What is occurring in this case?

(c) Define

$$w_1 := x + y + u, \quad w_2 := x^2 + y^2 + u^2, \quad w_3 = xy + xu + yu.$$

Show that $w_1(w_2 - w_3)$ is constant along the characteristic curves.