4.1. Conservation laws and critical times Consider the PDE

$$u_y + \partial_x(f(u)) = 0\,.$$

In the following cases, compute the critical time y_c (i.e., the first time when the solution becomes nonsmooth):

(a) $f(u) = \frac{1}{2}u^2$, the initial datum is $u(x,0) = \sin(x)$.

(b) $f(u) = \sin(u)$, the initial datum is $u(x,0) = \frac{1}{2}x^2$.

(c) $f(u) = e^u$, the initial datum is $u(x,0) = x^5$.

4.2. Multiple choice Cross the correct answer(s).

(a) In all generality, a conservation law (as we defined it in the lecture)

- ◯ admits a strong local solution
- ◯ admits a strong global solution
- ◯ develops singularities
- ◯ has finite critical time
- ◯ might have several weak solutions
- ◯ has straight lines as characteristics

(b) Consider the conservation law

$$\begin{cases} u_y + (\alpha u^2 - u)u_x = 0, & (x,y) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 1, & x < 0, \\ u(x,0) = 0, & x \geq 0. \end{cases}$$

Then, the shock wave solution has strictly positive slope if

- ◯ $\alpha > 1$
- ◯ $\alpha < 1$
- ◯ $\alpha > 0$
- ◯ $\alpha < 0$
- ◯ $\alpha > \frac{3}{2}$
- ◯ $\alpha < \frac{3}{2}$

(c) The conservation law

$$\begin{cases} u_y + (u^2 + 5)u_x = 0, & (x,y) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 1, & x < 0, \\ u(x,0) = \sqrt{1-x}, & x \in [0,1], \\ u(x,0) = 0, & x > 1, \end{cases} \tag{1}$$

has a crossing of characteristics at[1]

[1] You can partially check your answer computing y_c. Why?

- ◯ $(6,2)$
- ◯ $(1,2)$
- ◯ $(6,1)$
- ◯ $(0,1)$

(d) The shock wave solution of Equation (1) has slope

- ◯ $\frac{16}{3}$
- ◯ $\frac{31}{6}$
- ◯ $\frac{2}{6}$
- ◯ 0

Extra exercises

4.3. Weak solutions Consider the PDE

$$\partial_y u + \partial_x\left(\frac{u^4}{4}\right) = 0$$

in the region $x \in \mathbb{R}$ and $y > 0$.

(a) Show that the function $u(x,y) := \sqrt[3]{\frac{x}{y}}$ is a classical solution of the PDE.

(b) Show that the function

$$u(x,y) := \begin{cases} 0 & \text{if } x > 0, \\ \sqrt[3]{\frac{x}{y}} & \text{if } x \leq 0. \end{cases}$$

is a *weak* solution of the PDE.