5.1. Weak solutions

Consider the transport equation

$$u_y + \frac{1}{2}\partial_x\left(u^2\right) = 0. \tag{1}$$

(a) Suppose that u is a classical solution to the previous transport equation. What equation does u^2 fulfil? Write it in the form

$$v_y + \partial_x\left(F(v)\right) = 0, \tag{2}$$

for some appropriate F.

(b) Consider the weak solution of Equation (1) given by

$$w(x,y) = \begin{cases} 3 & \text{if } x < \frac{3}{2}y - 1 \\ 0 & \text{if } x > \frac{3}{2}y - 1. \end{cases}$$

Show that w^2 is not a weak solution of (2). Can you explain what is the problem?

5.2. Balance laws A generalization of the conservation law are the so called *balance laws*

$$\begin{cases} u_y + (f(u,x,y))_x = g(u,x,y), \\ u(x,0) = h(x). \end{cases}$$

Recalling that here $y > 0$ represents the time variable, the above PDE models the flow of mass with concentration $u(x,y)$ associated to a flux depending on the density, the time and the space. The term g represents the source term of the system.

(a) Consider the *transport equation with source term*: $f(u,x,y) = cu$, $c > 0$, and $g(u,x,y) = 2y$. Find a solution. Do the same for a general time dependent source term $g = g(y)$.

(b) Consider the modified Burger's equation in which the flux increases linearly in time: $f(u,x,y) = \frac{u^2}{2}y$, $g \equiv 0$. Set the initial condition $h(x) = 1$ if $x < 0$, $h(x) = 1 - x$ if $x \in [0,1]$ and $h(x) = 0$ if $x > 1$. Find a solution. What is the main difference with the solution of the Burger's equation ($f = \frac{u^2}{2}$)? [1]

[1] Try to sketch the characteristic curves

5.3. Multiple choice Cross the correct answer(s).

(a) The second order linear PDE given by

$$u_x + x^2 u_{xx} + 2x\sin(y)u_{xy} - \cos^2(y)u_{yy} + e^x = 0,$$

is

- ◯ hyperbolic if $x \neq 0$
- ◯ parabolic in $\{y = k\frac{\pi}{2} : k \in \mathbb{Z}\}$
- ◯ everywhere hyperbolic
- ◯ parabolic in $x = 0$

(b) Let $A = (a_{ij})$ be a (2×2) real matrix, and $f : \mathbb{R}^2 \to \mathbb{R}$ any smooth function. Then, the PDE [2]

$$\text{Trace}\Big(A \cdot D^2 u\Big) = f,$$

is

- ◯ elliptic if A is symmetric and $\det(A) > 0$
- ◯ hyperbolic if A is antisymmetric and $a_{11} = a_{22} \neq 0$
- ◯ parabolic if A is symmetric and $\det(A) < 0$
- ◯ hyperbolic if A is antisymmetric and $a_{11} = -a_{22} > 0$

(c) The same options as point **(b)**, but with the PDE

$$\text{div}\Big(A \cdot \nabla u\Big) = f.$$

(d) The following conservation law

$$\begin{cases} u_y + f(u)_x = 0, \\ u(x,0) = c > 0 \text{ for } x < 0 \text{ and } u(x,0) = 0 \text{ for } x \geq 0, \end{cases}$$

has a shock curve of slope equal to 8 if

- ◯ $c = 2$ and $f(u) = u^4$
- ◯ $c = 2$ and $f(u) = -u^4$
- ◯ $c = 1$ and $f(u) = u^3$
- ◯ $c = 1$ and $f(u) = 2u^2 + 6u - 1$

[2] Recall the definition of the Hessian matrix $(D^2u)_{ij} = u_{x_i x_j}$. To start simple: what does it happen when A is the identity matrix?

Extra exercises

5.4. Weak solutions II

Consider the equation

$$e^{-u}u_x + u_y = 0,$$

with initial value $u(x,0) = 0$ if $x < 0$, and $u(x,0) = \alpha > 0$ if $x > 0$.

(a) Find a weak solution for any $\alpha > 0$ with a single discontinuity for $y \geq 0$.

(b) Show that such solution fulfils the entropy condition for all $\alpha > 0$.

5.5. Finding shock waves

Consider the transport equation

$$u_y + u^2 u_x = 0,$$

with initial condition $u(x,0) = 1$ for $x \leq 0$, $u(x,0) = 0$ for $x \geq 1$, and

$$u(x,0) = \sqrt{1-x} \quad \text{for} \quad 0 < x < 1.$$

(a) Find the solution using the method of characteristics. Up to which time is the solution defined in a classical sense?

(b) Find a weak solution for all times $y \geq 0$.