6.1. Wave equation Consider the homogeneous one dimensional wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = 2\sin(x) + \cos(x), & x \in \mathbb{R}, \\ u_t(x,0) = 2, & x \in \mathbb{R}. \end{cases}$$

Compute the explicit solution u.

6.2. Wave equation's anatomy Consider the general homogeneous one dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

(a) Identify the backward and forward waves F and G. Give the necessary and sufficient condition on f and g to have $F = G$, or $F = -G$.

(b) Suppose that $f(x)$ and $g(x)$ are trigonometric polynomials of the form

$$f(x) = \sum_{n=0}^{N} a_n \cos(nx),$$
$$g(x) = \sum_{n=0}^{M} b_n \cos(nx),$$

where $\{a_n\}_{n=0}^N, \{b_n\}_{n=0}^M \subset \mathbb{R}$, and $N, M \geq 0$. Find a general solution u.

6.3. Propagation of symmetries from initial data

Consider the general wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0, \infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= g(x), & x \in \mathbb{R}. \end{cases}$$

(a) Suppose that both f and g are odd functions (that is, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$). Show that the solution u is also an odd function in x, for each time $t > 0$ (that is, $u(-x,t) = -u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$).

(b) Suppose now that both f and g are even (that is $f(x) = f(-x)$ and $g(x) = g(-x)$ for all $x \in \mathbb{R}$) and 2π-periodic functions. Using Exercise 6.2 (b), justify formally why $u(\cdot, t)$ has to be an even function.[1]

(c) Find an explicit example in which f is even, g is odd, and the solution is for $t > 0$ neither even nor odd.

6.4. Multiple choice Cross the correct answer(s).

(a) The second order linear PDE given by

$$u_x + x^2 u_{xx} + 2x\sin(y)u_{xy} - \cos^2(y)u_{yy} + e^x = 0,$$

is

- ◯ hyperbolic if $x \neq 0$
- ◯ everywhere hyperbolic
- ◯ parabolic in $\{y = k\frac{\pi}{2} : k \in \mathbb{Z}\}$
- ◯ parabolic in $x = 0$

(b) Consider u solution of the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = \cos(t), & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = x, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Then,

- ◯ for all $x \in \mathbb{R}$ one has that $\lim_{t\to+\infty} \frac{u(x,t)}{t} = 0$
- ◯ u is odd in x
- ◯ $u(x,t)$ is periodic in t
- ◯ u is periodic in x
- ◯ u is odd in t

(c) Consider u solution of the one dimensional wave equation

$$\begin{cases} u_{tt} - \pi^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = x^2, & x \in \mathbb{R}, \\ u_t(x,0) = -\sin(x), & x \in \mathbb{R}. \end{cases}$$

The value of u at $(x,t) = (\pi, 2)$ is equal to

[1] One can actually argue as in point (a) without needing the trigonometric expansions of exercise 6.2 (b). The purpose of this point is to see that the Fourier expansion gives another tool to investigate the symmetries of u.

- ◯ $5\pi^2$
- ◯ $3\pi^2$
- ◯ 0
- ◯ 2π

Extra exercises

6.5. Time reversible

Consider the Cauchy problem posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Let $\tilde{u}(x,t) := u(x,-t)$. Show that $\tilde{u}(x,t)$ solves the Cauchy problem posed for $-\infty < x < \infty$ and $t < 0$,

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0, & (x,t) \in \mathbb{R} \times (-\infty,0), \\ \tilde{u}(x,0) = f(x), & x \in \mathbb{R}, \\ \tilde{u}_t(x,0) = -g(x), & x \in \mathbb{R}. \end{cases}$$

That is, we are showing that the wave equation is reversible in time. If a function solves a wave equation, the same function with time reversed also solves a the wave equation with the same initial condition and opposite initial velocity.

6.6. Zero boundary condition

Use the previous exercise to solve the following Cauchy problem posed for $x > 0$ and $t > 0$, with zero boundary condition at $x = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,\infty) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(x,0) = x^4, & x \in (0,\infty), \\ u_t(x,0) = 0, & x \in (0,\infty). \end{cases}$$