7.1. (Non)homogeneous wave equation

(a) Let $u = u(x,t)$ be a solution of the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 1, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 1, & x \in \mathbb{R}, \\ u_t(x,0) = 2, & x \in \mathbb{R}, \end{cases}$$

Compute the explicit solution.

(b) Let $u = u(x,t)$ be a solution of the wave equation

$$\begin{cases} u_{tt} - 2u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = \sin(x), & x \in \mathbb{R}, \end{cases}$$

where $f(x) = x$, if $|x| \le 2$ and $f(x) = 0$, if $|x| > 2$. Is u smooth? Otherwise, where are the singularities of u? Compute the explicit solution *after* answering these questions.

7.2. Propagation of symmetries from initial data, II Consider the general nonhomogeneous wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= F(x,t), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= g(x), & x \in \mathbb{R}. \end{cases}$$

Take advantage of the uniqueness Theorem (4.4.6) in the notes to show that

(a) if f, g and $F(\cdot,t)$ are odd/even functions, then $u(\cdot,t)$ is itself odd/even.

(b) if f, g and $F(\cdot,t)$ are periodic with same period $T > 0$ (i.e. $f(x+T) = f(x)$, $g(x+T) = g(x)$ and $F(x+T,t) = F(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$), then $u(\cdot,t)$ is itself periodic with period T.

7.3. Wave equation on a ring Let $u : [0,1] \times [0,\infty) \to \mathbb{R}$ be a solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in [0,1] \times (0,\infty), \\ u(x,0) &= x - x^2, & x \in [0,1], \\ u_t(x,0) &= 0, & x \in [0,1], \\ u(0,t) &= u(1,t), & t \in (0,\infty), \\ u_x(0,t) &= u_x(1,t), & t \in (0,\infty). \end{cases}$$

Compute $u(1/2, 2023)$.

7.4. Multiple choice Cross the correct answer(s).

(a) Let u be solution of the homogeneous wave equation

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Let h be a smooth function, and u_h be the solution of the above PDE with perturbed initial condition $u_h(x,0) = f(x)$ and $(u_h)_t(x,0) = g(x)+h(x)$. Then, $u(1,2) = u_h(1,2)$

- ◯ whenever h has compact support in $[-5,7]$
- ◯ only when h constantly equal to zero
- ◯ whenever $\int_{-5}^{7} h(x)\,dx = 0$
- ◯ whenever h us equal to zero in $[-5,7]$

(b) Same question as (a), but when we perturb $u_h(x,0) = f(x) + h(x)$, $(u_h)_t(x,0) = g(x)$.

- ◯ only when h constantly equal to zero
- ◯ always for h small enough
- ◯ when $h(x) = \sin(\pi(x+1))$
- ◯ whenever $h(-5) = h(7) = 0$

(c) Let u be solution of the homogeneous wave equation

$$\begin{cases} u_{tt} - u_{xx} = F(x), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Suppose that F, f and g are trigonometric polynomials as in Exercise 6.2 (b), with $\int_0^{2\pi} g\,dx = 0$. Then, u is

- ◯ never
- ◯ always for $F \equiv 0$
- ◯ always
- ◯ never unless $F \neq f$

2π-periodic *in time*[1], that is $u(x,t+2\pi) = u(x,t)$ for all $(x,t) \in \mathbb{R} \times (0,+\infty)$.

[1] Be careful, this is not the same as being periodic in the x variable, as in Exercise 7.2.

Extra exercises

7.5. Strange wave equation Show that the following partial differential equation admits a solution

$$\begin{cases} u_{tt} - u_{xx} &= \frac{u_t^2 - u_x^2}{2u}, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= x^4, & x \in \mathbb{R}, \\ u_t(x,0) &= 0, & x \in \mathbb{R}\,. \end{cases}$$

Hint: Consider the function $v(x,t) = \sqrt{u(x,t)}$. *What equation does it satisfy?*