8.1. Separation of variables

Solve the following equations using the method of separation of variables and superposition principle. To do so, write first a general solution solving the problem with boundary conditions, and then impose the initial values.

(a)

$$\begin{cases} u_t - u_{xx} &= 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(2x) + 2\sin(3x) + 4\sin(4x), & x \in [0,\pi]. \end{cases}$$

(b)

$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= 2\sin^3(x), & x \in [0,\pi], \\ u_t(x,0) &= \sin(4x), & x \in [0,\pi]. \end{cases}$$

Hint: recall that $4\sin^3(x) = 3\sin(x) - \sin(3x)$.

(c)

$$\begin{cases} u_t - u_{xx} &= 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u_x(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= 1 + \cos(x) & x \in [0,\pi]. \end{cases}$$

8.2. Multiple choice Cross the correct answer(s).

(a) Let u be solution of the heat equation

$$\begin{cases} u_t - ku_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & x \in (0,L). \end{cases}$$

for $f \in C^\infty(0,T)$. Then, for all $a > 0$

- ○ $\lim_{t\to+\infty} \int_0^L u(x,t)^2\,dx = +\infty$
- ○ $\lim_{t\to+\infty} t^a \int_0^L u(x,t)^2\,dx = 0$
- ○ $\lim_{t\to+\infty} \int_0^L u(x,t)^2\,dx = 0$
- ○ $\lim_{t\to+\infty} t^a \int_0^L u(x,t)^2\,dx = +\infty$

(b) Consider the periodic homogeneous wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t) \in [0,1] \times [0,+\infty) \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = 1 + 2021\cos(2\pi x), & x \in [0,1], \\ u_t(x,0) = \cos(40\pi x), & x \in [0,1]. \end{cases}$$

Then, for a fixed point $\bar{x} \in [0,1]$, the function $t \mapsto u(\bar{x},t)$ has period

- ○ $1/40$
- ○ $1/2$
- ○ 2π
- ○ π