9.1. Separation of variables for non-homogeneous problems

Solve the following equations using the method of separation of variables and superposition principle. If the boundary conditions are non-homogeneous, find a suitable function satisfying the boundary conditions, and subtract it from the solution.

(a)

$$\begin{cases} u_t - u_{xx} &= t + 2\cos(2x), & (x,t) \in (0,\pi/2) \times (0,\infty), \\ u_x(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi/2,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= 1 + 2\cos(6x), & x \in [0,\pi/2]. \end{cases}$$

(b)

$$\begin{cases} u_t - u_{xx} &= 1 + x\cos(t), & (x,t) \in (0,1) \times (0,\infty), \\ u_x(0,t) &= \sin(t), & t \in (0,\infty), \\ u_x(1,t) &= \sin(t), & t \in (0,\infty), \\ u(x,0) &= 1 + \cos(2\pi x), & x \in [0,1]. \end{cases}$$

Hint: The function $w(x,t) = x\sin(t)$ fulfills the boundary conditions from above.

(c) Mixed Boundary Conditions.

$$\begin{cases} u_t - u_{xx} &= \sin(9x/2), & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(3x/2), & x \in [0,\pi]. \end{cases}$$

9.2. Conservation of energy Suppose $u(x,t)$ is periodic on $(0,\pi)$ and solves $u_{tt} - u_{xx} = f(x)$, for some periodic function f.

(a) Show that if $f \equiv 0$, then the energy

$$E(t) := \frac{1}{2}\int_0^\pi (u_t(x,t))^2 + (u_x(x,t))^2\,dx,$$

is conserved, i.e. $E(t) = E(0)$ for all $t > 0$.

(b) Inspired by the homogeneous case, find a similar conserved quantity when $f(x) = \sum_{n=1}^M A_n \sin(nx)$.

9.3. Multiple choice Cross the correct answer(s).

(a) Consider the non-homogeneous heat equation

$$\begin{cases} u_t - u_{xx} = p(t)u, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = u(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = f(x), & x \in (0,\pi), \end{cases}$$

where $p(t)$ is a given function of t, and $f(x) = \sum_{n=1}^{M} A_n \sin(nx) \not\equiv 0$. Then

- ◯ If $p(t) = (k+1)t^k$, $\lim_{t\to+\infty} e^{-t^{k+1}} u(1,t) = 0$.
- ◯ If $p(t) = (k+1)t^k$, $\lim_{t\to+\infty} u(1,t) = +\infty$.
- ◯ If $p(t) = \sin(t)$, $\lim_{t\to+\infty} u(1,t) = +\infty$.
- ◯ If $p(t) = 1$, $\lim_{t\to+\infty} u(1,t) = A_1 \sin(1)$.

Extra exercises

9.4. Solve the following non-homogeneous problem.

$$\begin{cases} u_t - u_{xx} &= -u, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(x), & x \in [0,\pi]. \end{cases}$$