10.1. Unique solution

Let $k > 0$. Let D be a bounded planar domain in $\mathbb{R}^2$. Let $u = u(x, y)$ be a solution to the Dirichlet problem for the reduced Helmholtz energy in D. That is, let u solve

$$\begin{cases} \Delta u(x,y) - ku(x,y) = 0, & \text{for } (x,y) \in D, \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial D. \end{cases}$$

Show that there exists at most a unique solution twice differentiable in D and continuous in $\overline{D}$, that is, $u \in C^2(D) \cap C(\overline{D})$.

Hint: Assume that there exist two solutions u_1 and u_2, and consider the difference $v = u_1 - u_2$.

10.2. The mean-value principle Let D be a planar domain, and let $B_R((x_\circ, y_\circ))$ (ball of radius R centered at $(x_\circ, y_\circ)$) be fully contained in D. Let u be an harmonic function in D, $\Delta u = 0$ in D. Then, the mean-value principle says that the value of u at $(x_\circ, y_\circ)$ is the average value of u on $\partial B_R((x_\circ, y_\circ))$. That is,

$$u(x_\circ, y_\circ) = \frac{1}{2\pi R} \oint_{\partial B_R((x_\circ, y_\circ))} u(x(s), y(s))\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\, d\theta.$$

Show that $u(x_\circ, y_\circ)$ is also equal to the average of u in $B_R((x_\circ, y_\circ))$, that is,

$$u(x_\circ, y_\circ) = \frac{1}{\pi R^2} \int_{B_R((x_\circ, y_\circ))} u(x, y)\, dx\, dy.$$

10.3. Maximum principle Consider the disk $D := \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\right\}$. Let $u = u(x, y)$ be a function twice differentiable in D and continuous in $\bar{D}$, solving

$$\begin{cases} \Delta u(x,y) = 0, & \text{in } D, \\ u(x,y) = g(x,y), & \text{on } \partial D, \end{cases}$$

for some given function g.

(a) Suppose $g(x, y) = x^2 + \frac{2}{\sqrt{2}} y$. Compute $u(0, 0)$ and $\max_{(x,y)\in\bar{D}} u(x, y)$.

(b) Suppose now that g is any smooth function such that $g(x, y) \geq (3x - y)$. Show that $u(1/3, 0) \geq 1$, with equality if and only if $g(x, y) = 3x - y$.

Hint: the function $3x - y$ is harmonic.

10.4. Multiple choice Cross the correct answer(s).

(a) Consider the Neumann problem for the Poisson equation

$$\begin{cases} \Delta u = \rho, & \text{in } D, \\ \partial_\nu u = g, & \text{on } \partial D, \end{cases}$$

where $D = B(0, R)$ is the ball of radius $R > 0$ with centre in the origin of $\mathbb{R}^2$, and ρ and g are given in polar coordinates (r, θ) by

$$\rho(r, \theta) = r^\alpha \sin^2(\theta), \text{ and } g(r, \theta) = C \cos^2(\theta) + r^{2021} \sin(\theta),$$

for some constants $\alpha > 0$ and $C > 0$. For which values of $C > 0$ does the problem satisfy the Neumann's *necessary* condition for existence of solutions?

- ○ $C = \frac{R^{\alpha+1}}{\alpha+2}$
- ○ $C = \frac{R^{\alpha+2}}{\alpha+2}$
- ○ $C = \frac{R^{\alpha+1}}{\alpha+1}$
- ○ $C = \frac{R^{\alpha+1}}{\alpha-1}$

(b) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \frac{x}{x^2+y^2} & \text{on } \partial D, \end{cases}$$

where the domain D is the anulus defined by $D := \left\{(x, y) \in \mathbb{R}^2 : 1 < \sqrt{x^2 + y^2} < 2\right\}$. What is the maximum of u?

- ○ $\frac{1}{2}$
- ○ $\frac{1}{4}$
- ○ 1
- ○ -1

Extra exercises

10.5. Weak maximum principle Let B_1 denote the unit ball in $\mathbb{R}^2$ centered at the origin, and let $u = u(x, y)$ be twice differentiable in B_1 and continuous in $\overline{B_1}$. Suppose that u solves the Dirichlet problem

$$\begin{cases} \Delta u(x, y) = -1, & \text{for } (x, y) \in B_1, \\ u(x, y) = g(x, y), & \text{for } (x, y) \in \partial B_1. \end{cases}$$

Show that

$$\max_{\bar{B}_1} u \leq \frac{1}{2} + \max_{\partial B_1} g.$$

Hint: search for a simple *function w such that $\Delta w = 1$, and use it to reduce the problem to an application of the weak maximum principle for harmonic functions.*

10.6. The mean-value principle II *(Hard)* Let u be an harmonic function in $\mathbb{R}^2$, $\Delta u = 0$ in $\mathbb{R}^2$. Use the result in Exercise 10.2 to show that for any smooth, radial, compactly-supported $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with $\int_{\mathbb{R}^2} \varphi(x, y)\, dx\, dy = 1$, we have

$$u(x_\circ, y_\circ) = \int_{\mathbb{R}^2} u(x, y)\varphi(x_\circ - x, y_\circ - y)\, dx\, dy.$$

(Harder) Use the above to show that u must be infinitely differentiable.