11.1. Separation of variables for elliptic equations

(a) Find a solution to

$$\begin{cases} \Delta u = 0, & (x,y) \in [0,\pi]^2, \\ u(x,0) = u(x,\pi) = 0, & x \in [0,\pi], \\ u(0,y) = 0, & y \in [0,\pi], \\ u(\pi,y) = \sin(2y), & y \in [0,\pi]. \end{cases}$$

(b) Find a solution to

$$\begin{cases} \Delta u = \sin(x) + \sin(3y), & (x,y) \in [\pi,2\pi]^2, \\ u(x,\pi) = 0, & x \in [\pi,2\pi], \\ u(x,2\pi) = -\sin(x), & x \in [\pi,2\pi], \\ u(\pi,y) = 0, & y \in [\pi,2\pi], \\ u(2\pi,y) = -\sin(3y)/9, & y \in [\pi,2\pi]. \end{cases}$$

Hint: find a simple function $f(x,y)$ such that $v := u + f$ is harmonic. Then, solve for v.

11.2. Heat Equation Let $u : [0,1] \times [0,+\infty) \to \mathbb{R}$ be solution of the heat equation

$$\begin{cases} u_y - u_{xx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ u(x,0) = x(1-x), & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in [0,+\infty). \end{cases}$$

Show that $0 \le u(0.5, 100) \le 0.00001$.

Hint: notice that $w(x,t) = e^{-\pi^2 t}\sin(\pi x)$ solves the same PDE with different initial conditions.

11.3. Uniqueness of solutions Let $D \subset \mathbb{R}^2$ be a planar domain and $f : \partial D \to \mathbb{R}$ a continuous function defined on its boundary. Show that the following elliptic problem

$$\begin{cases} \Delta u = u, & \text{in } D, \\ u = f, & \text{on } \partial D, \end{cases}$$

admits at most one smooth solution.

If u_1 and u_2 solve the same PDE, what can we say about $u_1 - u_2$?