12.1. Necessary condition

Let us consider the following problem for $0 < x, y < \pi$,

$$\begin{cases} \Delta u &= 0, & \text{for } 0 < x, y < \pi, \\ u_y(x,\pi) &= x^2 - a, & \text{for } 0 < x < \pi, \\ u_y(x,0) &= a - x^2, & \text{for } 0 < x < \pi, \\ u_x(0,y) = u_x(\pi,y) &= 0, & \text{for } 0 < y < \pi. \end{cases}$$

Find all the values of $a \in \mathbb{R}$ for which the problem admits a solution.

12.2. Separation of variables

Find the solution to the following problem, posed for $0 < x < 2\pi$ and $-1 < y < 1$.

$$\begin{cases} \Delta u &= 0, & \text{for } 0 < x < 2\pi, -1 < y < 1, \\ u(x,-1) &= 0, & \text{for } 0 \leq x \leq 2\pi, \\ u(x,1) &= 1 + \cos(2x), & \text{for } 0 \leq x \leq 2\pi, \\ u_x(0,y) = u_x(2\pi,y) &= 0, & \text{for } -1 < y < 1. \end{cases}$$

In order to do that, first find nontrivial solutions $w(x,y) = X(x)Y(y)$ to the following problem,

$$\begin{cases} \Delta w &= 0, & \text{for } 0 < x < 2\pi, -1 < y < 1, \\ w_x(0,y) = w_x(2\pi,y) &= 0, & \text{for } -1 < y < 1, \end{cases}$$

and use them as a base to generate the solution to the previous problem.

12.3. Neumann problem

Consider the Neumann boundary problem for the Laplace equation, for $0 < x, y < \pi$:

$$\begin{cases} \Delta u &= 0, & \text{for } 0 < x, y < \pi, \\ u_x(0,y) &= 0, & \text{for } 0 < y < \pi, \\ u_x(\pi,y) &= \sin(y), & \text{for } 0 < y < \pi, \\ u_y(x,0) &= 0, & \text{for } 0 < x < \pi, \\ u_y(x,\pi) &= -\sin(x), & \text{for } 0 < x < \pi. \end{cases}$$

(a) Show that this problem admits a solution.

(b) We now want to split the problem into two different problems such that the Neumann condition is zero in opposite sides. In order to do that, though, we need to make sure that the arising problems can still be solved (namely, the outward flux must be zero).

For that, let us consider the function $v = u + a(x^2 - y^2)$, for some $a \in \mathbb{R}$ to be determined. That is, we have subtracted an harmonic polynomial such that the flow of the normal derivative in opposite sides is non-zero. What is the problem solved by v? Write it in terms of the constant $a \in \mathbb{R}$.

(c) Split the problem for v into two different problems with zero Neumann conditions on opposite sides of the domain. Determine the value of $a \in \mathbb{R}$ for which such problems can be solved.

12.4. Laplace operator and rotations

For any $\theta \in [0, 2\pi]$ let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plan by θ radians given by the matrix

$$R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(a) Show that the Laplace operator $\Delta : u \mapsto u_{xx} + u_{yy}$ is *invariant* under the change of variables $(s, t) := R_\theta \cdot (x, y)$ in the following sense: expressing $u = u(x, y) \in C^2$ in the varaibles (s, t) as $v(s, t) := u(x(s, t), y(s, t))$, we have that

$$\begin{aligned} \Delta v(s,t) = v_{ss}(s,t) + v_{tt}(s,t) &= u_{xx}(x(s,t), y(s,t)) + u_{yy}(x(s,t), y(s,t)) \\ &= \Delta u(x(s,t), y(s,t)). \end{aligned}$$

(b) Use the previous point to conclude that now we are able to solve the Laplace equation $\Delta u = 0$ in any arbitrary rectangle of $\mathbb{R}^2$ with Dirichlet/Neumann boundary conditions (under the usual compatibility/smoothness assumptions).