10.1. Unique solution

Let $k > 0$. Let D be a bounded planar domain in $\mathbb{R}^2$. Let $u = u(x, y)$ be a solution to the Dirichlet problem for the reduced Helmholtz energy in D. That is, let u solve

$$\begin{cases} \Delta u(x,y) - ku(x,y) &= 0, & \text{for } (x,y) \in D, \\ u(x,y) &= g(x,y), & \text{for } (x,y) \in \partial D. \end{cases}$$

Show that there exists at most a unique solution twice differentiable in D and continuous in $\overline{D}$, that is, $u \in C^2(D) \cap C(\overline{D})$.

Hint: Assume that there exist two solutions u_1 and u_2, and consider the difference $v = u_1 - u_2$.

SOL:

Let us use the hint. Let us suppose that there exist two solutions u_1 and u_2 fulfilling the Dirichlet problem. Let $v = u_1 - u_2$. Notice that $v = v(x, y)$ solves

$$\begin{cases} \Delta v(x,y) - kv(x,y) &= 0, & \text{for } (x,y) \in D, \\ v(x,y) &= 0, & \text{for } (x,y) \in \partial D. \end{cases}$$

We just need to prove that $v \equiv 0$ in D. To do so, we will show that $\max_{\overline{D}} v = \min_{\overline{D}} v = 0$. We show both equalities by contradiction.

Notice that $\max_{\overline{D}} v \geq 0$, since $v = 0$ on ∂D. Let us suppose that $\max_{\overline{D}} v = M > 0$. In particular, there exists some $(x_\circ, y_\circ) \in D$ such that $v(x_\circ, y_\circ) = M > 0$, that is, v has a maximum at $(x_\circ, y_\circ)$. In particular, we know that $\Delta v(x_\circ, y_\circ) \leq 0$. Therefore,

$$0 = \Delta v(x_\circ, y_\circ) - kv(x_\circ, y_\circ) \leq -kM < 0,$$

a contradiction.

On the other hand, $\min_{\overline{D}} v \leq 0$, since $v = 0$ on ∂D. Let us suppose that $\min_{\overline{D}} v = m < 0$. In particular, there exists some $(x_\circ, y_\circ) \in D$ such that $v(x_\circ, y_\circ) = m < 0$, that is, v has a minimum at $(x_\circ, y_\circ)$. In particular, we know that $\Delta v(x_\circ, y_\circ) \geq 0$. Therefore,

$$0 = \Delta v(x_\circ, y_\circ) - kv(x_\circ, y_\circ) \geq -km > 0,$$

a contradiction. Therefore, if there exists a solution, is unique.

10.2. The mean-value principle Let D be a planar domain, and let $B_R((x_\circ, y_\circ))$ (ball of radius R centered at $(x_\circ, y_\circ)$) be fully contained in D. Let u be an harmonic function in D, $\Delta u = 0$ in D. Then, the mean-value principle says that the value of u at $(x_\circ, y_\circ)$ is the average value of u on $\partial B_R((x_\circ, y_\circ))$. That is,

$$u(x_\circ, y_\circ) = \frac{1}{2\pi R} \oint_{\partial B_R((x_\circ, y_\circ))} u(x(s), y(s))\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\, d\theta.$$

Show that $u(x_\circ, y_\circ)$ is also equal to the average of u in $B_R((x_\circ, y_\circ))$, that is,

$$u(x_\circ, y_\circ) = \frac{1}{\pi R^2} \int_{B_R((x_\circ, y_\circ))} u(x, y)\, dx\, dy.$$

SOL:

Let us use polar coordinates to compute

$$\begin{aligned}
\frac{1}{\pi R^2} \int_{B_R((x_\circ, y_\circ))} u(x, y)\, dx\, dy &= \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta) r\, d\theta\, dr \\
&= \frac{1}{\pi R^2} \int_0^R r \left(\int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta)\, d\theta \right) dr \\
&= \frac{1}{\pi R^2} \int_0^R 2\pi r u(x_\circ, y_\circ)\, dr \\
&= u(x_\circ, y_\circ) \frac{1}{\pi R^2} [\pi r^2]_0^R \\
&= u(x_\circ, y_\circ).
\end{aligned}$$

We have used here the boundary mean value principle in the balls $B_r((x_\circ, y_\circ))$ for each $r \in (0, R)$.

10.3. Maximum principle Consider the disk $D := \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\right\}$. Let $u = u(x, y)$ be a function twice differentiable in D and continuous in $\bar{D}$, solving

$$\begin{cases} \Delta u(x, y) = 0, & \text{in } D, \\ u(x, y) = g(x, y), & \text{on } \partial D, \end{cases}$$

for some given function g.

(a) Suppose $g(x, y) = x^2 + \frac{2}{\sqrt{2}} y$. Compute $u(0, 0)$ and $\max_{(x,y)\in\bar{D}} u(x, y)$.

(b) Suppose now that g is any smooth function such that $g(x, y) \geq (3x - y)$. Show that $u(1/3, 0) \geq 1$, with equality if and only if $g(x, y) = 3x - y$.

Hint: the function $3x - y$ is harmonic.

SOL:

(a) By the mean value property

$$u(0,0) = \frac{1}{2\pi}\int_{\partial D} g\,dl = \frac{1}{2\pi}\int_0^{2\pi} \cos(\theta)^2 + \frac{2}{\sqrt{2}}\sin(\theta)\,d\theta = \frac{\pi}{2\pi} = \frac{1}{2}.$$

By the Maximum Principle

$$\max_{\bar{D}} u = \max_{\partial D} u = \max_{\partial D} g = \max_{\theta\in[0,2\pi)} \{\cos(\theta)^2 + \frac{2}{\sqrt{2}}\sin(\theta)\}.$$

Setting $g(\theta) = \cos^2(\theta) + \frac{2}{\sqrt{2}}\sin(\theta)$, we have that (up to periodicity)

$$g'(\theta) = \cos(\theta)(\frac{2}{\sqrt{2}} - 2\sin(\theta)) = 0,$$

if and only if $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}\}$. A quick check shows that $\max_\theta g(\theta) = g(\pi/4) = \frac{3}{2}$.

(b) It is convenient to set the auxiliary function $w := u - 3x + y$. Then

$$\begin{cases} \Delta w = 0, & \text{in } D \\ w = g - (3x - y) \geq 0, & \text{on } \partial D, \end{cases}$$

by the very assumption on g. Applying the Maximum Principle to w, we get that

$$\min_{\bar{D}}(u - (3x - y)) = \min_{\bar{D}} w = \min_{\partial D} w \geq 0,$$

implying that $u(x, y) \geq 3x - y$ in $\bar{D}$. In particular,

$$u(1/3, 0) \geq 3 \cdot \frac{1}{3} = 1.$$

If $u(1/3, 0) = 1$, then w attains its minimum in D since $w(1/3, 0) = u(1/3, 0) - 1 = 0$. This implies by the strong maximum principle that $w \equiv 0$, and hence $u(x, y) = 3x - y$. In particular, $g(x, y) = u(x, y) = 3x - y$ on ∂D. This shows the 'only if' direction.

The 'if' direction is a consequence of uniqueness of solution of the Laplace equations with Dirichlet boundary condition on w. More precisely, if $g = 3x - y$, then this solves the boundary condition and it is itself harmonic in the D. Since solutions of Laplace's eqn. are uniquely determined by their boundary conditions, we must have that $g = 3x - y$ being the unique solution and we see that $u(1/3, 0) = 1$.

10.4. Multiple choice Cross the correct answer(s).

(a) Consider the Neumann problem for the Poisson equation

$$\begin{cases} \Delta u = \rho, & \text{in } D, \\ \partial_\nu u = g, & \text{on } \partial D, \end{cases}$$

where $D = B(0, R)$ is the ball of radius $R > 0$ with centre in the origin of $\mathbb{R}^2$, and ρ and g are given in polar coordinates (r, θ) by

$$\rho(r, \theta) = r^\alpha \sin^2(\theta), \text{ and } g(r, \theta) = C \cos^2(\theta) + r^{2021} \sin(\theta),$$

for some constants $\alpha > 0$ and $C > 0$. For which values of $C > 0$ does the problem satisfy the Neumann's *necessary* condition for existence of solutions?

- X $C = \frac{R^{\alpha+1}}{\alpha+2}$
- ◯ $C = \frac{R^{\alpha+2}}{\alpha+2}$
- ◯ $C = \frac{R^{\alpha+1}}{\alpha+1}$
- ◯ $C = \frac{R^{\alpha+1}}{\alpha-1}$

SOL: We say that the Neumann Problem for the Poisson equation satisfies the necessary condition for existence of solutions if the identity

$$\int_{\partial D} g = \int_D \rho, \tag{1}$$

holds. In our particular case we can compute in polar coordinates

$$\int_D \rho = \int_0^R r \int_0^{2\pi} r^\alpha \sin^2(\theta)\, d\theta\, dr = \pi \frac{R^{\alpha+2}}{\alpha + 2},$$

and parametrizing ∂D with the curve $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ we have that

$$\int_{\partial D} g = \int_0^{2\pi} R\Big(C\cos^2(\theta) + R^{2021}\sin(\theta)\Big)\, d\theta = RC\pi.$$

Plugging this in Equation (1) we obtain that the identity

$$RC\pi = \pi \frac{R^{\alpha+2}}{\alpha + 2},$$

is valid if and only if $C = \frac{R^{\alpha+1}}{\alpha+2}$.

(b) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \frac{x}{x^2+y^2} & \text{on } \partial D, \end{cases}$$

where the domain D is the anulus defined by $D := \Big\{(x, y) \in \mathbb{R}^2 : 1 < \sqrt{x^2 + y^2} < 2\Big\}$. What is the maximum of u?

○ $\frac{1}{2}$ ○ $\frac{1}{4}$

X 1 ○ -1

SOL: By the weak maximum principle, $\max_{\bar{D}} u = \max_{\partial D} u = \max_{\partial D} \frac{x}{x^2+y^2}$. Writing $\partial D = \{x^2+y^2=1\} \cup \{x^2+y^2=4\} =: S^1 \cup S^2$, we check that $\max_{S^1} u = \max_{S^1} x = 1$, and $\max_{S^2} u = \max_{S^2} \frac{x}{4} = \frac{1}{2}$. Hence $\max_{\partial D} u = \max\{1, \frac{1}{2}\} = 1$.

Extra exercises

10.5. Weak maximum principle Let B_1 denote the unit ball in $\mathbb{R}^2$ centered at the origin, and let $u = u(x,y)$ be twice differentiable in B_1 and continuous in $\overline{B_1}$. Suppose that u solves the Dirichlet problem

$$\begin{cases} \Delta u(x,y) &= -1, & \text{for } (x,y) \in B_1, \\ u(x,y) &= g(x,y), & \text{for } (x,y) \in \partial B_1. \end{cases}$$

Show that

$$\max_{\bar{B}_1} u \leq \frac{1}{2} + \max_{\partial B_1} g.$$

Hint: search for a simple *function w such that $\Delta w = 1$, and use it to reduce the problem to an application of the weak maximum principle for harmonic functions.*

SOL:

We just need to find a function $w(x,y)$ such that $\Delta w(x,y) = 1$, and then consider $v(x,y) = u(x,y) + w(x,y)$. The simplest function such that $\Delta w(x,y) = 1$ is $w(x,y) = \frac{1}{2}x^2$. Thus, let us define

$$v(x,y) = u(x,y) + \frac{1}{2}x^2.$$

Then, v solves

$$\begin{cases} \Delta v(x,y) &= 0, & \text{for } (x,y) \in B_1, \\ v(x,y) &= g(x,y) + \frac{1}{2}x^2, & \text{for } (x,y) \in \partial B_1. \end{cases}$$

By the weak maximum principle, we know that

$$\max_{\bar{B}_1} v(x,y) = \max_{\partial B_1} \left(g(x,y) + \frac{1}{2}x^2 \right) \leq \max_{\partial B_1} g(x,y) + \max_{\partial B_1} \frac{1}{2}x^2.$$

Notice that $\max_{\partial B_1} \frac{1}{2}x^2 = \frac{1}{2}$, so

$$\max_{\bar{B}_1} v(x,y) \leq \frac{1}{2} + \max_{\partial B_1} g(x,y).$$

On the other hand, $v(x,y) \geq u(x,y)$ for all $x, y \in B_1$, so

$$\max_{\bar{B}_1} u(x,y) \leq \max_{\bar{B}_1} v(x,y) \leq \frac{1}{2} + \max_{\partial B_1} g(x,y),$$

as we wanted to see.

10.6. The mean-value principle II *(Hard)* Let u be an harmonic function in $\mathbb{R}^2$, $\Delta u = 0$ in $\mathbb{R}^2$. Use the result in Exercise 10.2 to show that for any smooth, radial, compactly-supported $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with $\int_{\mathbb{R}^2} \varphi(x,y)\, dx\, dy = 1$, we have

$$u(x_\circ, y_\circ) = \int_{\mathbb{R}^2} u(x,y)\varphi(x_\circ - x, y_\circ - y)\, dx\, dy.$$

(Harder) Use the above to show that u must be infinitely differentiable.

SOL:

Due to radial symmetry, we can write $\varphi(x,y) = \varphi(R)$ in polar coordinates. Now the condition on φ translates to

$$\int_0^\infty \varphi(R) R\, dR = 1\,.$$

Now we have

$$\begin{aligned}
\int_{\mathbb{R}^2} u(x,y)\varphi(x_\circ - x, y_\circ - y)\, dxdy &= \int_{\mathbb{R}^2} u(x_\circ + x, y_\circ + y)\varphi(-x,-y)\, dxdy \\
&= \int_0^\infty \int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\varphi(R)\, d\theta R dR \\
&= \int_0^\infty \varphi(R) R \int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\, d\theta dR \\
&= \int_0^\infty \varphi(R) R u(x_\circ, y_\circ)\, dR = u(x_\circ, y_\circ)\,.
\end{aligned}$$

We used a change of variable and radial symmetry of φ in the first line and the mean value priniciple of Exercise 10.2 in the fourth line.

To show that u is infinitely-differentiable, let us suppose that u is just C^k for some $2 \leq k \in \mathbb{N}$. So $\partial_x^\alpha \partial_y^\beta u$ exists and is continuous for $\alpha + \beta \leq k$. So, by the above we have

$$\begin{aligned}
\partial_x^\alpha \partial_y^\beta u(x_\circ, y_\circ) &= \partial_x^\alpha \partial_y^\beta \int_{\mathbb{R}^2} u(x,y)\varphi(x_\circ - x, y_\circ - y)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \partial_x^\alpha \partial_y^\beta u(x,y)\varphi(x_\circ - x, y_\circ - y)\, dx\, dy \\
&= (-1)^{\alpha+\beta} \int_{\mathbb{R}^2} u(x,y)\partial_x^\alpha \partial_y^\beta \varphi(x_\circ - x, y_\circ - y)\, dx\, dy
\end{aligned}$$

where in the third line we integrated by parts (here, we used that φ is compactly-supported to make sure the integrals are finite). But now we see that

$$(-1)^{\alpha+\beta} \int_{\mathbb{R}^2} \partial_x u(x,y) \partial_x^\alpha \partial_y^\beta \varphi(x_\circ - x, y_\circ - y)\, dx\, dy$$

is a well-defined expression for $\partial_x \partial_x^\alpha \partial_y^\beta u$ and satisfies the usual limit definition of taking a derivative. Similarly for ∂_y. Thus, we see that $u \in C^{k+1}$. But this contradicts that u is just C^k. Thus, u must be infinitely-differentiable.