11.1. Separation of variables for elliptic equations

(a) Find a solution to

$$\begin{cases} \Delta u = 0, & (x,y) \in [0,\pi]^2, \\ u(x,0) = u(x,\pi) = 0, & x \in [0,\pi], \\ u(0,y) = 0, & y \in [0,\pi], \\ u(\pi,y) = \sin(2y), & y \in [0,\pi]. \end{cases}$$

(b) Find a solution to

$$\begin{cases} \Delta u = \sin(x) + \sin(3y), & (x,y) \in [\pi,2\pi]^2, \\ u(x,\pi) = 0, & x \in [\pi,2\pi], \\ u(x,2\pi) = -\sin(x), & x \in [\pi,2\pi], \\ u(\pi,y) = 0, & y \in [\pi,2\pi], \\ u(2\pi,y) = -\sin(3y)/9, & y \in [\pi,2\pi]. \end{cases}$$

Hint: find a simple function $f(x,y)$ such that $v := u + f$ is harmonic. Then, solve for v.

SOL:

(a) We are looking for an harmonic function u such that

$$u(0,y) = 0, u(\pi,y) = \sin(2y), u(x,0) = u(x,\pi) = 0.$$

(Notice that since the solution at $(x,0)$ and (x,π) is already 0, we do not need to split it into two functions, and we can directly work with u. Compare with Example 7.21 in Pinchover's.)

We use separation of variables, and we assume that u can be expressed as sum of harmonic functions $w(x,y) = X(x)Y(y)$. Imposing that w is harmonic we reach that

$$Y''(y) + \lambda Y(y) = 0,$$

and $Y(0) = Y(\pi) = 0$. On the other hand, we also reach

$$X''(x) - \lambda X(x) = 0.$$

The problem for Y is already standard, and we have as eigenvalues $\lambda_n = n^2$ and as eigenfucntions $Y_n(y) = \sin(ny)$, for $n = 1, 2, \ldots$. Thus, the equation for X becomes simply

$$X_n''(x) - n^2 X_n(x) = 0.$$

Solutions to the previous problem are of the form $X_n(x) = \alpha_n \sinh(nx) + \beta_n \cosh(nx)$. However, such basis (in terms of $\sinh(nx)$ and $\cosh(nx)$) is not very useful when dealing with boundary behaviour for this problem at $x = 0, \pi$. Thus, we choose instead the basis $\sinh(nx)$ and $\sinh(nx - n\pi)$. Let us now show why we can express the solution in that basis. That is, we want to write

$$X_n(x) = \gamma_n \sinh(nx) + \delta_n \sinh(n(x - \pi)),$$

and find the coefficients γ_n and δ_n in terms of α_n and β_n. To do that, we use that $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$, that sinh is odd and cosh is even. Therefore,

$$\sinh(n(x - \pi)) = \sinh(nx)\cosh(n\pi) - \cosh(nx)\sinh(n\pi),$$

and

$$X_n(x) = (\gamma_n + \delta_n \cosh(n\pi))\sinh(nx) - \delta_n \sinh(n\pi)\cosh(nx),$$

and we get that $\beta_n = -\delta_n \sinh(n\pi)$ and $\alpha_n = \gamma_n + \delta_n \cosh(n\pi)$. That is, $\delta_n = -\beta_n / \sinh(n\pi)$ and $\gamma_n = \alpha_n - \delta_n \cosh(n\pi)$; and both bases are interchangeable.

Thus, let us express the solution $u(x, y)$ as

$$u(x, y) = \sum_{n \geq 1} \sin(ny) \left(\delta_n \sinh(nx) + \gamma_n \sinh(n(x - \pi))\right).$$

Now, since $u(0, y) = 0$, we deduce that $\gamma_n = 0$. On the other hand, since $u(\pi, y) = \sin(2y)$,

$$u(\pi, y) = \sum_{n \geq 1} \delta_n \sin(ny) \sinh(n\pi) = \sin(2y),$$

we deduce that $\delta_2 = \frac{1}{\sinh(2\pi)}$, and $\delta_n = 0$ for $n \neq 2$. Thus, our solution is going to be

$$u(x, y) = \sin(2y) \frac{\sinh(x)}{\sinh(2\pi)}.$$

(b) Since $\sin'' = -\sin$, we can easily check that

$$0 = \Delta u - \sin(x) - \sin(3y) = \Delta\left(u + \sin(x) + \frac{\sin(3y)}{9}\right).$$

Hence, setting $v(x, y) := u(x, y) + \sin(x) + \frac{\sin(3y)}{9}$ we obtain that v is an harmonic function solving

$$\begin{cases} \Delta v = 0, & \text{for } \pi < x < 2\pi, \pi < y < 2\pi, \\ v(x, \pi) = \sin(x), & \text{for } \pi \leq x \leq 2\pi, \\ v(x, 2\pi) = 0, & \text{for } \pi \leq x \leq 2\pi, \\ v(\pi, y) = \frac{\sin(3y)}{9}, & \text{for } \pi \leq y \leq 2\pi, \\ v(2\pi, y) = 0, & \text{for } \pi \leq y \leq 2\pi. \end{cases}$$

We factorize then $v = v_1 + v_2$ where

$$\begin{cases} \Delta v_1 &= 0, & \text{for } \pi < x < 2\pi, \pi < y < 2\pi, \\ v_1(x,\pi) &= 0, & \text{for } \pi \leq x \leq 2\pi, \\ v_1(x,2\pi) &= 0, & \text{for } \pi \leq x \leq 2\pi, \\ v_1(\pi,y) &= \frac{\sin(3y)}{9}, & \text{for } \pi \leq y \leq 2\pi, \\ v_1(2\pi,y) &= 0, & \text{for } \pi \leq y \leq 2\pi. \end{cases}$$

and

$$\begin{cases} \Delta v_2 &= 0, & \text{for } \pi < x < 2\pi, \pi < y < 2\pi, \\ v_2(x,\pi) &= \sin(x), & \text{for } \pi \leq x \leq 2\pi, \\ v_2(x,2\pi) &= 0, & \text{for } \pi \leq x \leq 2\pi, \\ v_2(\pi,y) &= 0, & \text{for } \pi \leq y \leq 2\pi, \\ v_2(2\pi,y) &= 0, & \text{for } \pi \leq y \leq 2\pi. \end{cases}$$

This corresponds to the following splitting:

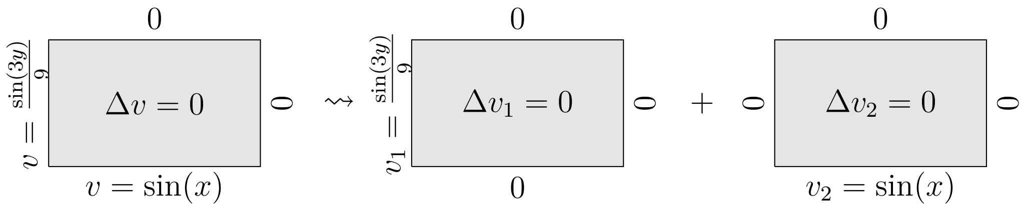

Figure 1: Splitting of the Laplace equation.

After operating the classical separation of variable, we have that

$$v_1(x,y) = \sum_{n=1}^{+\infty} \Big(A_n \sinh(n(x-\pi)) + B_n \sinh(n(x-2\pi)) \Big) \sin(n(y-\pi)),$$
$$v_2(x,y) = \sum_{n=1}^{+\infty} \Big(C_n \sinh(n(y-\pi)) + D_n \sinh(n(y-2\pi)) \Big) \sin(n(x-\pi)),.$$

To determinate the coefficients, we have to take advantage of the boundary data:

$$0 = v_1(2\pi, y) = \sum_{n=1}^{+\infty} A_n \sinh(n\pi) \sin(n(y-\pi)),$$

which implies $A_n \equiv 0$. On the other side,

$$\frac{\sin(3y)}{9} = v_1(\pi, y) = \sum_{n=1}^{+\infty} B_n \sinh(-n\pi) \sin(n(y-\pi)),$$

and since $\sin(3y) = \sin(3(y-\pi))$ we obtain that $B_2 = (9\sinh(-3\pi))^{-1} = -(9\sinh(3\pi))^{-1}$, and $B_n = 0$ otherwise. Similarly, $C_n \equiv 0$ and combining

$$\sin(x) = v_2(x,\pi) = \sum_{n=1}^{+\infty} D_n \sinh(-\pi n)\sin(n(x-\pi)),$$

with the identity $\sin(x) = -\sin(x-\pi)$ we obtain that $D_1 = (-\sinh(-\pi))^{-1} = \sinh(\pi)^{-1}$, and $D_n = 0$ otherwise. Combining everything we obtain that

$$\begin{aligned} u(x,y) &= v(x,y) - \sin(x) - \frac{\sin(3y)}{9} = v_1(x,y) + v_2(x,y) - \sin(x) - \frac{\sin(3y)}{9} \\ &= -\frac{\sinh(3(x-2\pi))}{9\sinh(3\pi)}\sin(3(y-\pi)) + \frac{\sinh(y-2\pi)}{\sinh(\pi)}\sin(x-\pi) - \sin(x) - \frac{\sin(3y)}{9} \\ &= -\Big(\frac{\sinh(3(x-2\pi))}{9\sinh(3\pi)} + \frac{1}{9}\Big)\sin(3y) - \Big(\frac{\sinh(y-2\pi)}{\sinh(\pi)} + 1\Big)\sin(x). \end{aligned}$$

11.2. Heat Equation Let $u : [0,1] \times [0,+\infty) \to \mathbb{R}$ be solution of the heat equation

$$\begin{cases} u_y - u_{xx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ u(x,0) = x(1-x), & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in [0,+\infty). \end{cases}$$

Show that $0 \le u(0.5, 100) \le 0.00001$.

Hint: notice that $w(x,t) = e^{-\pi^2 t}\sin(\pi x)$ solves the same PDE with different initial conditions.

SOL: First, let us check that $w := e^{-\pi^2 t}\sin(\pi x)$ solves the equation:

$$\partial_t w = -\pi^2 w = \partial_{xx} w\,.$$

One can check that $\sin(\pi x) \ge x(1-x)$ in the interval $[0,1]$ (for example, you can use wolfram-alpha for checking this!). Thus, we have $w(x,0) \ge u(x,0)$.

Similarly, 0 solves the equation and $u(x,0) \ge 0$.

By the comparison principle for solutions of the heat equation (which is an easy consequence of the maximum principle for parabolic equations), we deduce $0 \le u(x,t) \le w(x,t)$ for all $x \in (0,1)$ and $t \ge 0$. In particular we find

$$0 \le u(0.5,100) \le w(0.5,100) = e^{-100\pi^2}\sin\Big(\frac{\pi}{2}\Big) = e^{-100\pi^2} \ll 0.00001\,.$$

11.3. Uniqueness of solutions Let $D \subset \mathbb{R}^2$ be a planar domain and $f : \partial D \to \mathbb{R}$ a continuous function defined on its boundary. Show that the following elliptic problem

$$\begin{cases} \Delta u = u, & \text{in } D, \\ u = f, & \text{on } \partial D, \end{cases}$$

admits at most one smooth solution.

If u_1 and u_2 solve the same PDE, what can we say about $u_1 - u_2$?

SOL: Let $u_1, u_2 : \bar{D} \to \mathbb{R}$ be two solutions. Let $v := u_1 - u_2$ be the difference. Notice that v satisfies

$$\begin{cases} \Delta v = v, & \text{in } D, \\ v = 0, & \text{on } \partial D, \end{cases}$$

Assume that $v > 0$ somewhere in D. Let $(x, y) \in D$ be the maximum point of v. Then we have $v(x, y) > 0$ and $\Delta v \leq 0$, which is a contradiction since $v = \Delta v$.

Hence, it must be $v \leq 0$. Similarly (just repeating the argument for $-v$ instead of v) we can show $v \geq 0$. Hence $v = 0$ everywhere and thus $u_1 = u_2$.