12.1. Necessary condition

Let us consider the following problem for $0 < x, y < \pi$,

$$\begin{cases} \Delta u &= 0, & \text{for } 0 < x, y < \pi, \\ u_y(x,\pi) &= x^2 - a, & \text{for } 0 < x < \pi, \\ u_y(x,0) &= a - x^2, & \text{for } 0 < x < \pi, \\ u_x(0,y) = u_x(\pi,y) &= 0, & \text{for } 0 < y < \pi. \end{cases}$$

Find all the values of $a \in \mathbb{R}$ for which the problem admits a solution.

Sol. We recall that, since u is harmonic in the square $0 < x, y < \pi$, then, from the Green identities, in order to have a solution the flux of such solution must be equal to zero. Namely, if we denote Q our domain, and ∂Q is its boundary, then the problem will have a solution if

$$\int_{\partial Q} \partial_n u \, dS = 0,$$

where $\partial_n u$ denotes the outward normal derivative of u. (See equation (7.9) from the book.)

In this case, we should have

$$\begin{aligned} 0 &= \int_{\partial Q} \partial_n u \, dS \\ &= \int_0^\pi u_y(x,\pi)\, dx + \int_0^\pi u_x(\pi,y)\, dy - \int_0^\pi u_y(x,0)\, dx - \int_0^\pi u_x(0,y)\, dy \\ &= \int_0^\pi (x^2 - a)\, dx + 0 - \int_0^\pi (a - x^2)\, dx - 0 \\ &= 2\int_0^\pi (x^2 - a)\, dx \\ &= \frac{2}{3}\pi^3 - 2a\pi. \end{aligned}$$

Thus, we must have $a = \frac{1}{3}\pi^2$ for the problem to have a solution. (In fact, once the problem has a solution, adding any constant gives another solution, so the problem has infinitely many solutions.)

12.2. Separation of variables

Find the solution to the following problem, posed for $0 < x < 2\pi$ and $-1 < y < 1$.

$$\begin{cases} \Delta u &= 0, & \text{for } 0 < x < 2\pi, -1 < y < 1, \\ u(x,-1) &= 0, & \text{for } 0 \le x \le 2\pi, \\ u(x,1) &= 1 + \cos(2x), & \text{for } 0 \le x \le 2\pi, \\ u_x(0,y) = u_x(2\pi,y) &= 0, & \text{for } -1 < y < 1. \end{cases}$$

In order to do that, first find nontrivial solutions $w(x, y) = X(x)Y(y)$ to the following problem,

$$\begin{cases} \Delta w = 0, & \text{for } 0 < x < 2\pi, -1 < y < 1, \\ w_x(0, y) = w_x(2\pi, y) = 0, & \text{for } -1 < y < 1, \end{cases}$$

and use them as a basis to generate the solution to the previous problem.

Sol. Let us proceed by separation of variables. Since $\Delta w = 0$, then

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda,$$

for some $\lambda \in \mathbb{R}$. Thus, we know

$$\begin{aligned} X''(x) + \lambda X(x) = 0, &\quad \text{for} \quad 0 < x < 2\pi, \\ Y''(y) - \lambda Y(y) = 0, &\quad \text{for} \quad -1 < y < 1. \end{aligned}$$

On the other hand, from the boundary condition we must have that $X'(0) = X'(2\pi) = 0$. As it is already standard, the solutions to the eigenvalue problem for X are given by

$$X_n(x) = \cos\left(\frac{nx}{2}\right), \qquad \lambda_n = \left(\frac{n}{2}\right)^2,$$

for $n = 0, 1, 2, \ldots$. Similarly, solving for Y using the values of λ_n we have just found, we reach that

$$Y_0(y) = \alpha_0 y + \beta_0,$$

and for $n \geq 1$,

$$Y_n(y) = \alpha_n \sinh\left(\frac{n(y+1)}{2}\right) + \beta_n \sinh\left(\frac{n(y-1)}{2}\right).$$

Notice that we have chosen the basis into which express Y_n so that, when trying to impose the conditions at $y = -1, 1$, we get a simpler expression. That is, instead of considering the standard basis given by $\sinh(ny/2)$ and $\cosh(ny/2)$ we have shifted it so to make is simpler in the following steps. (See the previous exercise sheet for a discussion on why this can be done.)

Thus, our solutions $w_n(x, y)$ are of the form:

$$\begin{aligned} &w_0(x, y) = \alpha_0 y + \beta_0, \\ &w_n(x, y) = \cos\left(\frac{nx}{2}\right)\left(\alpha_n \sinh\left(\frac{n(y+1)}{2}\right) + \beta_n \sinh\left(\frac{n(y-1)}{2}\right)\right), \end{aligned}$$

for $n = 1, 2, 3, \ldots$.

We now use the superposition principle, to say that we are looking for a solution to our problem of the form

$$\begin{aligned} u(x,y) &= w_0(x,y) + w_1(x,y) + w_2(x,y) + \ldots \\ &= \alpha_0 y + \beta_0 + \sum_{n\geq 1} \cos\left(\frac{nx}{2}\right)\left(\alpha_n \sinh\left(\frac{n(y+1)}{2}\right) + \beta_n \sinh\left(\frac{n(y-1)}{2}\right)\right). \end{aligned}$$

Notice that, from the way we have built the functions w_n, such solution already fulfils the Neumann boundary conditions $u_x(0,y) = u_x(2\pi, y) = 0$ for $-1 < y < 1$. Moreover, since each w_n is harmonic, u is also harmonic. Thus, we just need to impose the other boundary conditions (the Dirichlet boundary conditions for this problem). Notice that, the special base we have chosen to express w_n, will be extremely useful in this step. Let us start imposing the boundary condition at $y = -1$:

$$u(x,-1) = -\alpha_0 + \beta_0 + \sum_{n\geq 1} \cos\left(\frac{nx}{2}\right)\beta_n \sinh(-n) = 0,$$

so that, since $\sinh(-n) \neq 0$, we get $\beta_n = 0$ for all $n \geq 1$, and $\beta_0 = \alpha_0$. On the other hand, imposing the boundary condition at $y = 1$,

$$u(x,1) = \alpha_0 + \beta_0 + \sum_{n\geq 1} \cos\left(\frac{nx}{2}\right)\alpha_n \sinh(n) = 1 + \cos(2x).$$

That is, on the one hand we get that $\alpha_0 + \beta_0 = 1$, so that, recalling that $\alpha_0 = \beta_0$ we get that $\alpha_0 = \beta_0 = \frac{1}{2}$. For $1 \leq n \neq 4$ we get that $\alpha_n \sinh(n) = 0$ (and therefore, $\alpha_n = 0$ if $1 \leq n \neq 4$), and $\alpha_4 \sinh(4) = 1$. That is, $\alpha_4 = \frac{1}{\sinh(4)}$.

Putting all together, we reach that our solution is

$$u(x,y) = \frac{1}{2}y + \frac{1}{2} + \frac{\cos(2x)}{\sinh(4)}\sinh(2(y+1)).$$

12.3. Neumann problem

Consider the Neumann boundary problem for the Laplace equation for $0 < x, y < \pi$:

$$\begin{cases} \Delta u = 0, & \text{for } 0 < x, y < \pi, \\ u_x(0,y) = 0, & \text{for } 0 < y < \pi, \\ u_x(\pi,y) = \sin(y), & \text{for } 0 < y < \pi, \\ u_y(x,0) = 0, & \text{for } 0 < x < \pi, \\ u_y(x,\pi) = -\sin(x), & \text{for } 0 < x < \pi. \end{cases}$$

(a) Show that this problem admits a solution.

Sol. As in exercise 11.1, we need to check that the outward flux of the solution (the integral of the normal derivative) vanishes:

$$\begin{aligned}\int_{\partial Q} \partial_n u \, dS &= \int_0^\pi u_y(x,\pi)\,dx + \int_0^\pi u_x(\pi,y)\,dy - \int_0^\pi u_y(x,0)\,dx - \int_0^\pi u_x(0,y)\,dy \\ &= -\int_0^\pi \sin(x)\,dx + \int_0^\pi \sin(y)\,dy - 0 - 0 \\ &= 0.\end{aligned}$$

Hence, the problem admits a solution (infinitely many up to adding a constant).

(b) We now want to split the problem into two different problems such that the Neumann condition is zero in opposite sides. In order to do that, though, we need to make sure that the arising problems can still be solved (namely, the outward flux must be zero).

For that, let us consider the function $v = u + a(x^2 - y^2)$, for some $a \in \mathbb{R}$ to be determined. That is, we have subtracted an harmonic polynomial such that the flow of the normal derivative in opposite sides is non-zero. What is the problem solved by v? Write it in terms of the constant $a \in \mathbb{R}$.

Sol. Notice that $v_x(0,y) = u_x(0,y) + a(x^2-y^2)_x|_{x=0} = 0 + 2ax|_{x=0} = 0$. Similarly, we have that $v_x(\pi,y) = u_x(\pi,y) + 2ax|_{x=\pi} = \sin(y) + 2a\pi$, $v_y(x,0) = 0$ and $v_y(x,\pi) = -\sin(x) - 2a\pi$. Thus, v solves the problem

$$\begin{cases} \Delta v = 0, & \text{for } 0 < x, y < \pi, \\ v_x(0,y) = 0, & \text{for } 0 < y < \pi, \\ v_x(\pi,y) = \sin(y) + 2a\pi, & \text{for } 0 < y < \pi, \\ v_y(x,0) = 0, & \text{for } 0 < x < \pi, \\ v_y(x,\pi) = -\sin(x) - 2a\pi, & \text{for } 0 < x < \pi. \end{cases}$$

(c) Split the problem for v into two different problems with zero Neumann conditions on opposite sides of the domain. Determine the value of $a \in \mathbb{R}$ for which such problems can be solved.

Sol. We split $v = v_1 + v_2$, where v_1 and v_2 solve the problems

$$\begin{cases} \Delta(v_1) = 0, & \text{for } 0 < x, y < \pi, \\ (v_1)_x(0,y) = 0, & \text{for } 0 < y < \pi, \\ (v_1)_x(\pi,y) = \sin(y) + 2a\pi, & \text{for } 0 < y < \pi, \\ (v_1)_y(x,0) = 0, & \text{for } 0 < x < \pi, \\ (v_1)_y(x,\pi) = 0, & \text{for } 0 < x < \pi. \end{cases}$$

and

$$\begin{cases} \Delta(v_2) &= 0, & \text{for } 0 < x, y < \pi, \\ (v_2)_x(0,y) &= 0, & \text{for } 0 < y < \pi, \\ (v_2)_x(\pi,y) &= 0, & \text{for } 0 < y < \pi, \\ (v_2)_y(x,0) &= 0, & \text{for } 0 < x < \pi, \\ (v_2)_y(x,\pi) &= -\sin(x) - 2a\pi, & \text{for } 0 < x < \pi. \end{cases}$$

The flux for each problem must be 0. In particular, we must have that for v_1,

$$\int_0^{\pi} (\sin(y) + 2a\pi)\, dy = 0 \quad \Rightarrow \quad 2a\pi^2 = -[-\cos(y)]_0^{\pi} = \quad \Rightarrow \quad a = \frac{-1}{\pi^2}.$$

We get the same result for v_2, either by the same computation, or by symmetry.

Thus, each of the previous two problems can be solved if $a = \frac{-1}{\pi^2}$.

12.4. Laplace operator and rotations

For any $\theta \in [0, 2\pi]$ let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plan by θ radians given by the matrix

$$R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(a) Show that the Laplace operator $\Delta : u \mapsto u_{xx} + u_{yy}$ is *invariant* under the change of variables $(s,t) := R_\theta \cdot (x,y)$ in the following sense: expressing $u = u(x,y) \in C^2$ in the varaibles (s,t) as $v(s,t) := u(x(s,t), y(s,t))$, we have that

$$\begin{aligned} \Delta v(s,t) = v_{ss}(s,t) + v_{tt}(s,t) &= u_{xx}(x(s,t), y(s,t)) + u_{yy}(x(s,t), y(s,t)) \\ &= \Delta u(x(s,t), y(s,t)). \end{aligned}$$

Sol. By the relation $(s,t) = R_\theta \cdot (x,y)$ we have that $(x,y) = R_\theta^{-1} \cdot (s,t) = R_{-\theta} \cdot (s,t)$, obtaining x and y in terms of s and t as

$$\begin{cases} x = x(s,t) &= \cos(\theta)s + \sin(\theta)t, \\ y = y(s,t) &= -\sin(\theta)s + \cos(\theta)t, \end{cases}$$

so that $v(s,t) = u\Big(\cos(\theta)s + \sin(\theta)t, -\sin(\theta)s + \cos(\theta)t\Big)$. The exercise reduces to apply multiple times the chain rule:

$$\begin{aligned} v_s(s,t) &= \Big(u\Big(x(s,t), y(s,t)\Big)\Big)_s \\ &= u_x\Big(x(s,t), y(s,t)\Big)x_s(s,t) + u_y\Big(x(s,t), y(s,t)\Big)y_s(s,t) \\ &= (\cos(\theta)s + \sin(\theta)t)_s u_x + (-\sin(\theta)s + \cos(\theta)t)_s u_y \\ &= \cos(\theta)u_x - \sin(\theta)u_y. \end{aligned}$$

Similarly

$$\begin{aligned}
v_s &= \cos(\theta)u_x - \sin(\theta)u_y, \\
v_t &= \sin(\theta)u_x + \cos(\theta)u_y, \\
v_{ss} &= \cos^2(\theta)u_{xx} - 2\sin(\theta)\cos(\theta)u_{xy} + \sin^2(\theta)u_{yy}, \\
v_{tt} &= \sin^2(\theta)u_{xx} + 2\sin(\theta)\cos(\theta)u_{xy} + \cos^2(\theta)u_{yy}.
\end{aligned}$$

It is straightforward that $v_{ss} + v_{tt} = u_{xx} + u_{yy}$.

(b) Use the previous point to conclude that now we are able to solve the Laplace equation $\Delta u = 0$ in any arbitrary rectangle of $\mathbb{R}^2$ with Dirichlet/Neumann boundary conditions (under the usual compatibility/smoothness assumptions).

Sol. If the rectangular domain has sides mutually parallel to the axis x and y, than we already developed the tools to solve the Laplace equation in Lectures 11 and 12. If the rectangle is rotated, it suffices (up to translation) to solve the problem rotating it back (and the boundary data accordingly), solve it, and then operate the change of variables of the previous point.