3.1. Characteristic method and initial conditions Consider the equation

$$xu_y - yu_x = 0.$$

For each of the following initial conditions, solve the problem in $y \geq 0$ whenever it is possible. If it is not, explain why.

(a) $u(x, 0) = x^2$.

(b) $u(x, 0) = x$.

(c) $u(x, 0) = x$ for $x > 0$.

SOL: In all three cases the initial condition is of the form $\Gamma(s) = \{s, 0, f(s)\}$ for some given function f. We can find an implicit solution via the method of characteristic solving the associated ODE system

$$\begin{cases} \frac{dx(t,s)}{dt} = -y(t,s), & x(0,s) = s, \\ \frac{dy(t,s)}{dt} = x(t,s) & y(0,s) = 0, \\ \frac{d\tilde{u}(t,s)}{dt} = 0, & \tilde{u}(t,s) = f(s). \end{cases}$$

By Exercise 2.1 (d), we know how to solve this system (recall: by differentiating once again in t to lose the crossed dependencies), obtaining $x(s,t) = s\cos(t)$, $y(s,t) = s\sin(t)$ and $\tilde{u}(s,t) = f(s)$. It follows that $s^2 = x(s,t)^2 + y(s,t)^2$, obtaining formally

$$u(x, y) = f\left(\pm\sqrt{x^2 + y^2}\right).$$

(a) Since in this case $f(s) = s^2$, $u(x, y) = x^2 + y^2$ without ambiguity in the sign.

(b) In this case we have no solution because if we chose $u(x, y) = \sqrt{x^2 + y^2}$ we have that $u(x, 0) = |x| \neq x$ for $x < 0$. Similarly, if we set $u(x, y) = -\sqrt{x^2 + y^2}$ then we have the exact same problem when $y = 0$ and $x > 0$. Geometrically, the reason of non existence is the following: notice that the characteristics curves $t \mapsto (x(t,s), y(t,s))$ are arcs of circles centered in the origin and crossing the x-axis in $(-s, 0)$ and $(s, 0)$, $s \geq 0$. The value of u along the characteristics is constant equal to $f(s)$. Hence, $f(-s) = u(-s, 0) = u(s, 0) = f(s)$, that is f has to be *even* for the solution to be well defined.

(c) In this case $f(s)$ is defined only when $s > 0$. By the discussion of the previous point, we need to impose $u(x, 0) = -x$ for all $x < 0$ to have well defined solution. Therefore we extend f by setting $f(s) = |s|$. In this case $u(x, y) = |\pm\sqrt{x^2 + y^2}| = |\sqrt{x^2 + y^2}|$ is well defined. Notice however that u is not C^1 at the origin, so u is a classical solution only in $\{(x, y) : x \neq 0, y \geq 0\}$.

3.2. Method of characteristic, local and global existence Consider the quasilinear, first order PDE

$$\begin{cases} u_x + \ln(u)u_y = u, & (x,y) \in \mathbb{R}^2, \\ u(x,0) = e^x, & x \in \mathbb{R}, \end{cases}$$

(here $\ln(\cdot)$ stands for the natural logarithm).

(a) Check the transversality condition.

(b) Find an explicit solution, and check if the result matches the existence condition found in the previous point.

SOL:

(a) The given PDE is of the form $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$ where $a(x,y,u) = 1$, $b(x,y,u) = \ln(u)$ and $c(x,y,u) = u$. The initial curve can be chosen equal to $\Gamma(s) = \{s, 0, e^s\}$. We compute the determinant

$$\det \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} \\ \frac{dx}{ds} & \frac{dy}{ds} \end{bmatrix}\Bigg|_{t=0} = \det \begin{bmatrix} 1 & \ln(e^s) \\ \frac{d}{ds}s & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & s \\ 1 & 0 \end{bmatrix} = -s.$$

The transversality condition is ensured provided $s \neq 0$.

(b) The ODE system is

$$\begin{cases} \frac{dx}{dt} = 1, & x(0,s) = s, \\ \frac{dy}{dt} = \ln(\tilde{u}), & y(0,s) = 0, \\ \frac{d\tilde{u}}{dt} = \tilde{u}, & \tilde{u}(0,s) = e^s. \end{cases}$$

Now, $x(t,s) = t + s$ and $\tilde{u}(t,s) = e^s e^t = e^{t+s}$. Plugging the solution for $\tilde{u}$ in the ODE for y we get $\frac{dy}{dt} = \ln(e^{t+s}) = t + s$, obtaining $y(t,s) = \frac{1}{2}t^2 + ts$. Normally we need to invert the map

$$(s,t) \mapsto (x(t,s), y(t,s)) = (t + s, t^2/2 + ts)).$$

But in this particular case, since $\tilde{u}(t,s) = e^{t+s}$, and $x = t + s$ we have immediately that $u(x,y) = e^x$. This solution is defined *globally*. This shows that the transversality condition is a sufficient condition for having a local solution, but tells us nothing about the possible existence of a global solution.

3.3. Multiple choice Cross the correct answer(s).

(a) Consider the first order linear PDE: $(x + e^y)u_x + u_y = x$. Then, the transversality condition is everywhere satisfied if

X $u(0,y) = y$

X $u(x,0) = \sin(x)$

◯ $u(x,x) = xy$

X $u(x^2,x) = 0$

SOL: From left to right:

$$\Gamma(s) = \{0,s,s\}, \quad \det \begin{bmatrix} e^s & 1 \\ 0 & 1 \end{bmatrix} = e^s,$$
$$\Gamma(s) = \{s,0,\sin(s)\}, \quad \det \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix} = -1,$$
$$\Gamma(s) = \{s,s,s^2\}, \quad \det \begin{bmatrix} s+e^s & 1 \\ 1 & 1 \end{bmatrix} = s + e^s - 1,$$
$$\Gamma(s) = \{s^2,s,0\}, \quad \det \begin{bmatrix} s^2+e^s & 1 \\ 2s & 1 \end{bmatrix} = s^2 + e^s - 2s.$$

The first two determinants are always different from zero. The third one is equal to zero when $s = 0$. The last one is always strictly greater than zero, since $e^s \geq s+1$ implies $s^2 + e^s - 2s \geq s^2 - s + 1 > 0$.

(b) Consider the first order quasilinear PDE: $xu_x + e^u u_y = 0$. Then, the transversality condition is satisfied if

X $u(x,x^2) = \ln(1+x^2)$, $x > 1$

◯ $u(x,x^2) = \ln(1+x^2)$, $x \geq 0$

◯ $u(0,y) = y$

X $u(x,0) = h(x)$ for any function h

SOL: From left to right:

$$\Gamma(s) = \{s,s^2,\ln(1+s^2)\}, s > 1, \quad \det \begin{bmatrix} s & 1+s^2 \\ 1 & 2s \end{bmatrix} = 2s^2 - 1 - s^2 = s^2 - 1 > 0,$$
$$\Gamma(s) = \{s,s^2,\ln(1+s^2)\}, s \geq 0, \quad \det \begin{bmatrix} s & 1+s^2 \\ 1 & 2s \end{bmatrix} = 2s^2 - 1 - s^2 = s^2 - 1,$$
$$\Gamma(s) = \{0,s,s\}, \quad \det \begin{bmatrix} 0 & e^s \\ 0 & 1 \end{bmatrix} = 0,$$
$$\Gamma(s) = \{s,0,h(s)\}, \quad \det \begin{bmatrix} s & e^{h(s)} \\ 1 & 0 \end{bmatrix} = -e^{h(s)} < 0.$$

Notice that the second determinant is equal to zero when $s = 1$.

(c) For which values of $r > 0$ there exists a local solution for

$$xu_x + (u+y)u_y = x^3 + 2,$$

in a neighbourhood of the circle $C_r := \{x^2 + y^2 = r^2\}$, so that $u|_{C_r} \equiv -1$?

X $r > 1$ ◯ $0 < r < 1$

◯ $r \geq 1$ ◯ $r = 1$

SOL: For a fixed $r > 0$ we parametrize the initial curve in polar coordinates $\Gamma(s) = \{r\cos(s), r\sin(s), -1\}$, $s \in [0, 2\pi)$. To check local existence in a neighbourhood of $\Gamma(s)$ we compute

$$\det \begin{bmatrix} r\cos(s) & -1 + r\sin(s) \\ -r\sin(s) & r\cos(s) \end{bmatrix} = r^2\cos(s)^2 + r\sin(s)(-1 + r\sin(s)) = r^2 - r\sin(s).$$

Notice that the above expression has always strictly positive maximum (since $r > 0$), and attains its minimum when $\sin(s) = 1$, so that we need to check when $r^2 - r > 0$. This is the case if and only if $r > 1$.

(d) For which values of $a > 0$ there exists a local solution of

$$uu_x + (y + a)u_y = 2022,$$

in a neighbourhood of the ellipse $E_a := \{\frac{x^2}{a^2} + y^2 = 1\}$, so that $u|_{E_a} = x$?

◯ $a = 1$ X $0 < a < 1$

◯ $a > 0$ ◯ $a \geq 1$

SOL: The solution is similar to point (c): for a fixed parameter $a > 0$ we parametrize the initial curve $\Gamma(s) = \{a\cos(s), \sin(s), a\cos(s)\}$. Then, in order to check the transversality condition we compute

$$\det \begin{bmatrix} a\cos(s) & \sin(s) + a \\ -a\sin(s) & \cos(s) \end{bmatrix} = a\cos(s)^2 + a\sin(s)^2 + a^2\sin(s) = a + a^2\sin(s).$$

This function in s has always strictly positive maximum (since $a > 0$) and reaches its minimum when $\sin(s) = -1$, that is at the value $a - a^2$, which is striclty positive if and only if $0 < a < 1$.

Extra exercises

3.4. Characteristic method and transversality condition Consider the transport equation

$$yu_x + uu_y = x.$$

(a) Solve the problem with initial condition $u(s, s) = -2s$, for $s \in \mathbb{R}$. For what domain of s does the transversality condition hold?

(b) Check the transversality condition with the initial value $u(s,s) = s$. What is occurring in this case?

(c) Define

$$w_1 := x + y + u, \quad w_2 := x^2 + y^2 + u^2, \quad w_3 = xy + xu + yu.$$

Show that $w_1(w_2 - w_3)$ is constant along the characteristic curves.

SOL:

(a) The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = y(t,s), && y_t(t,s) = \tilde{u}(t,s), && \tilde{u}_t(t,s) = x(t,s), \\ &x(0,s) = s, && y(0,s) = s, && u(0,s) = -2s. \end{aligned}$$

Notice that, if we define $w(t,s) := x(t,s) + y(t,s) + \tilde{u}(t,s)$, then $w_t(t,s) = w(t,s)$ and $w(0,s) = 0$. That is, $w(t,s) \equiv 0$ for all s, and therefore,

$$u(x,y) = -x - y.$$

Regarding the transversality condition, let us check:

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} y(0,s) & \tilde{u}(0,s) \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} s & -2s \\ 1 & 1 \end{vmatrix} = 3s \neq 0, \qquad \text{if} \qquad s \neq 0.$$

That is, the transversality condition holds if $s \neq 0$.

(b) The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = y(t,s), && y_t(t,s) = \tilde{u}(t,s), && \tilde{u}_t(t,s) = x(t,s), \\ &x(0,s) = s, && y(0,s) = s, && u(0,s) = s. \end{aligned}$$

Regarding the transversality condition, let us check:

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} y(0,s) & \tilde{u}(0,s) \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} s & s \\ 1 & 1 \end{vmatrix} = 0.$$

The transversality condition never holds. What is occurring is that the solution to the characteristic equations is (se^t, se^t, se^t), which coincides with the initial curve. In other words, from the PDE and the initial condition, we get no information on u outside of the line $s \mapsto (s,s,s)$.

Therefore, the problem is under-determined, and it has infinitely many solutions.

(c) characteristic curves fulfill the equations

$$x_t(t) = y(t), \quad y_t(t) = \tilde{u}(t), \quad \tilde{u}_t(t) = x(t)$$

(we removed the parameter s, since we will not care about initial value conditions for this part).

In particular, if we consider w_i along the curves, we can take $\tilde{w}_i(t) := w_i(x(t), y(t), \tilde{u}(t))$. We want to show that $\frac{d}{dt}\tilde{w}_1(\tilde{w}_2 - \tilde{w}_3) = 0$. Indeed:

$$\begin{aligned}
\frac{d\tilde{w}_1(t)}{dt} &= \tilde{w}_1(t), \\
\frac{d\tilde{w}_2(t)}{dt} &= 2x(t)u(t) + 2y(t)\tilde{u}(t) + 2x(t)y(t) = 2\tilde{w}_3(t), \\
\frac{d\tilde{w}_3(t)}{dt} &= y^2(t) + x(t)\tilde{u}(t) + x^2(t) + y(t)\tilde{u}(t) + \tilde{u}^2(t) + y(t)x(t) \\
&= \tilde{w}_2(t) + \tilde{w}_3(t).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{d}{dt}\tilde{w}_1(\tilde{w}_2 - \tilde{w}_3) &= \left(\frac{d}{dt}\tilde{w}_1\right)(\tilde{w}_2 - \tilde{w}_3) + \tilde{w}_1\left(\frac{d}{dt}\tilde{w}_2 - \frac{d}{dt}\tilde{w}_3\right) \\
&= \tilde{w}_1(\tilde{w}_2 - \tilde{w}_3) + \tilde{w}_1(2\tilde{w}_3 - \tilde{w}_2 - \tilde{w}_3) = 0,
\end{aligned}$$

as we wanted to see.