4.1. Conservation laws and critical times Consider the PDE

$$u_y + \partial_x(f(u)) = 0\,.$$

In the following cases, compute the critical time y_c (i.e., the first time when the solution becomes nonsmooth):

(a) $f(u) = \frac{1}{2}u^2$, the initial datum is $u(x,0) = \sin(x)$.

(b) $f(u) = \sin(u)$, the initial datum is $u(x,0) = \frac{1}{2}x^2$.

(c) $f(u) = e^u$, the initial datum is $u(x,0) = x^5$.

SOL: Recall the formula for the critical time

$$y_c := \inf\Big\{-\frac{1}{c'(u_0(s))u_0'(s)} : s \in \mathbb{R},\, c(u_0(s))_s < 0\Big\}.$$

(a) In this case $c(u) = f'(u) = u$ and $u_0(s) = \sin(s)$. Hence $c'(u_0(s))u_0'(s) = u_0'(s) = \cos(s)$, which is strictly negative then $s \in (\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi)$, $k \in \mathbb{Z}$. The critical time is then given by

$$y_c = \inf\Big\{-\cos(s)^{-1} : s \in (\frac{\pi}{2}, \frac{3\pi}{2})\Big\} = 1.$$

(b) In this case $c(u) = f'(u) = \cos(u)$ and $u_0(s) = \frac{1}{2}s^2$. Hence $c'(u_0(s))u_0'(s) = -\sin(\frac{1}{2}s^2)s$. Set $\tau := \frac{1}{2}s^2 \geq 0$, then

$$c'(u_0(s))u_0'(s) = -\sin(s^2/2)s = \begin{cases} \sin(\tau)\sqrt{2\tau}, & s \leq 0, \\ -\sin(\tau)\sqrt{2\tau}, & s > 0. \end{cases}$$

Since $\sqrt{\tau} \geq 0$, we have that $c'(u_0(s))u_0'(s) < 0$ when $s < 0$ and $\tau \in (\pi+2k\pi, 2\pi+2k\pi)$, or when $s > 0$ and $\tau \in (2k\pi, \pi + 2k\pi)$, $k \in \mathbb{Z}$. In particular, when $s > 0$, setting the sequence $\tau_k := \frac{\pi}{2} + 2k\pi$, one has that

$$(\sin(\tau_k)\sqrt{2\tau_k})^{-1} = (\sqrt{2\tau_k})^{-1} \to 0, \text{ when } k \to +\infty.$$

This shows that in this degenerate case $y_c = 0$. This means that the further we go along the initial curve $\Gamma(s)$, the sooner we see singularity forming near the x axis[1].

(c) In this case $c(u) = f'(u) = e^u$, and $u_0(s) = s^5$. Again, $c'(u_0(s))u_0'(s) = 5e^{s^5}s^4$. This quantity is always non negative, hence $y_c = \inf\emptyset = +\infty$. There is no formation of shock waves in this lucky case.

4.2. Multiple choice Cross the correct answer(s).

(a) In all generality, a conservation law (as we defined it in the lecture)

[1] Plot the function $(\sqrt{2x}\sin(x))^{-1}$ with your favourite software (Wolframalpha, Geogebra, etc) to convince yourself.

- X admits a strong local solution
- ◯ admits a strong global solution
- ◯ develops singularities
- ◯ has finite critical time
- X might have several weak solutions
- X has straight lines as characteristics

(b) Consider the conservation law

$$\begin{cases} u_y + (\alpha u^2 - u)u_x = 0, & (x,y) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 1, & x < 0, \\ u(x,0) = 0, & x \geq 0. \end{cases}$$

Then, the shock wave solution has strictly positive slope if

- ◯ $\alpha > 1$
- ◯ $\alpha < 1$
- ◯ $\alpha > 0$
- ◯ $\alpha < 0$
- X $\alpha > \frac{3}{2}$
- ◯ $\alpha < \frac{3}{2}$

SOL: The slope is computed via Rankine-Hugoniot formula

$$\sigma'(y) = \frac{f^+ - f^-}{u^+ - u^-}.$$

Here, $u^+ = 0$ and $u^- = 1$, and $f(u) = \alpha\frac{u^3}{3} - \frac{u^2}{2}$, so that $f^+ = f(u^+) = 0$ and $f^- = f(u^-) = \frac{\alpha}{3} - \frac{1}{2}$. This means that $\sigma'(y) = \frac{\alpha}{3} - \frac{1}{2}$, that is strictly positive if and only if $\alpha > \frac{3}{2}$.

(c) The conservation law

$$\begin{cases} u_y + (u^2 + 5)u_x = 0, & (x,y) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 1, & x < 0, \\ u(x,0) = \sqrt{1-x}, & x \in [0,1], \\ u(x,0) = 0, & x > 1, \end{cases} \tag{1}$$

has a crossing of characteristics at[2]

- ◯ $(6,2)$
- X $(6,1)$
- ◯ $(1,2)$
- ◯ $(0,1)$

[2] You can partially check your answer computing y_c. Why?

SOL: Setting the initial curve $\Gamma(s) = (s, 0, h(s))$, with $h(s) = 1$, $h(s) = \sqrt{1-s}$ and $h(s) = 0$, according to $s < 0$, $s \in [0,1]$ and $s > 1$, we get that $y(t,s) = t$, $u_0(s) = h(s)$ and $\frac{dx}{dt} = u_0(s)^2 + 5$, $x(0,s) = s$, so that $x(t,s) = t(h(s)^2 + 5) + s$. In particular, when $s < 0$, the characteristic curves are $(x(t,s), y(t,s)) = (6t + s, t)$, and when $s > 1$, $(x(t,s), y(t,s)) = (5t + s, t)$, and when $s \in [0,1]$ they are a collection of affine lines (given by $(x(t,s), y(t,s)) = (t(6-s) + s, t)$ which is obtained by plugging in $h(s) = \sqrt{1-s}$ in the formula for $x(t,s)$) passing through the same intersection point[3]. Therefore, we can restrict our attention computing the intersection between the curves $t \mapsto (6t, t)$ (last characteristic on the left), and $t \mapsto (5t + 1, t)$ (first characteristic on the right). Hence $6t = 5t + 1$ implies $t = 1$, and hence the intersection point is $(x, y) = (6, 1)$.

(d) The shock wave solution of Equation (1) has slope

- ◯ $\frac{16}{3}$
- ◯ $\frac{31}{6}$
- ◯ $\frac{2}{6}$
- ◯ 0

SOL: Applying Rankine-Hugoniot we have that $u^+ = 0$, $u^- = 1$ and $f(u) = \frac{u^3}{3} + 5u$, so that

$$\sigma'(u) = \frac{1}{3} + 5 = \frac{16}{3}.$$

Extra exercises

4.3. Weak solutions Consider the PDE

$$\partial_y u + \partial_x \left(\frac{u^4}{4} \right) = 0$$

in the region $x \in \mathbb{R}$ and $y > 0$.

(a) Show that the function $u(x,y) := \sqrt[3]{\frac{x}{y}}$ is a classical solution of the PDE.

(b) Show that the function

$$u(x,y) := \begin{cases} 0 & \text{if } x > 0, \\ \sqrt[3]{\frac{x}{y}} & \text{if } x \le 0. \end{cases}$$

is a *weak* solution of the PDE.

[3] Check this! Also look at Picture 3.5 in the Lecture Notes: the situation is similar, with $\alpha = 1$ and different slopes of the characteristic on the left and right side of the interval $[0,1]$.

SOL:

(a) Notice that $u_x = \frac{u}{3x}$ and $u_y = \frac{-u}{3y}$. Thus we have

$$u_y + \partial_x\left(\frac{u^4}{4}\right) = u_y + u_x u^3 = u_y + \frac{x}{y}u_x = \frac{-u}{3y} + \frac{x}{y}\frac{u}{3x} = 0\,.$$

(b) First of all, notice tha the function u is continuous.

Let us recall that a function u is a weak solution if for any $x_0 < x_1$ and any $0 < y_0 < y_1$, it holds

$$\int_{x_0}^{x_1} u(x, y_1) - u(x, y_0) + \int_{y_0}^{y_1} f(u(x_1, y)) - f(u(x_0, y)) = 0\,. \tag{2}$$

Since a classical solution is also a weak solution, thanks to what we have shown in part *(a)*, we already know that if $x_0 < x_1 \leq 0$, then (2) holds. Since also the constant 0 is a classical solution of the PDE, we have that (2) holds also if $0 \leq x_0 < x_1$.

It remains to prove the validity of (2) when $x_0 < 0 < x_1$. Thanks to what we have said above, we already know that (respectively setting $x_1 = 0$ and $x_0 = 0$)

$$\int_{x_0}^{0} u(x, y_1) - u(x, y_0) + \int_{y_0}^{y_1} f(u(0, y)) - f(u(x_0, y)) = 0\,,$$
$$\int_{0}^{x_1} u(x, y_1) - u(x, y_0) + \int_{y_0}^{y_1} f(u(x_1, y)) - f(u(0, y)) = 0\,.$$

Summing the two latter identities, we obtain exactly (2) for $x_0 < 0 < x_1$.