5.1. Weak solutions

Consider the transport equation

$$u_y + \frac{1}{2}\partial_x \left(u^2\right) = 0. \tag{1}$$

(a) Suppose that u is a classical solution to the previous transport equation. What equation does u^2 fulfil? Write it in the form

$$v_y + \partial_x \left(F(v)\right) = 0, \tag{2}$$

for some appropriate F.

(b) Consider the weak solution of Equation (1) given by

$$w(x,y) = \begin{cases} 3 & \text{if } x < \frac{3}{2}y - 1 \\ 0 & \text{if } x > \frac{3}{2}y - 1. \end{cases}$$

Show that w^2 is not a weak solution of (2). Can you explain what is the problem?

SOL:

(a) Let $v = u^2$. We first compute

$$v_y = \partial_y(u^2) = 2uu_y, \qquad v_x = 2uu_x.$$

This gives, using the equation for u, namely $u_y + uu_x = 0$,

$$v_y = 2uu_y = -2u^2 u_x = -uv_x = -v^{\frac{1}{2}} v_x,$$

and the last term can be rewritten as $\partial_x(F(v)) = F'(v)v_x$ if $F'(v) = v^{\frac{1}{2}}$, hence $F(v) = \frac{2}{3}v^{\frac{3}{2}}$.

(b) Let us denote

$$w(x,y) = \begin{cases} 3 & \text{if } x < \frac{3}{2}y - 1 \\ 0 & \text{if } x > \frac{3}{2}y - 1. \end{cases}$$

We have to check whether w^2 is a weak solution to

$$v_y + \frac{2}{3}\partial_x \left(v^{\frac{3}{2}}\right) = 0,$$

with $v(x,0) = 9$ for $x < -1$, and $v(x,0) = 0$ for $x > -1$. Notice that the curve of discontinuity of weak solutions to our Cauchy problem is given by is $x = \gamma(y)$ where

$$\gamma_y(y) = \frac{F(v^+) - F(v^-)}{v^+ - v^-} = \frac{2}{3}\frac{0 - 9^{\frac{3}{2}}}{0 - 9} = 2.$$

On the other hand, the curve of discontinuity of w^2 is of slope $\frac{3}{2}$. That is, w^2 cannot be a weak solution to (2).

What is happening is that to derive (2) from the equation $u_y + uu_x = 0$ we have applied the chain rule (for instance to say that $\partial_x(u^2) = 2uu_x$) which is rigorous only for smooth functions. This exercise actually shows that the chain rule is false for functions with jumps and one needs to be careful when dealing with weak solutions.

5.2. Balance laws A generalization of the conservation law are the so called *balance laws*

$$\begin{cases} u_y + (f(u,x,y))_x = g(u,x,y), \\ u(x,0) = h(x). \end{cases}$$

Recalling that here $y > 0$ represents the time variable, the above PDE models the flow of mass with concentration $u(x,y)$ associated to a flux depending on the density, the time and the space. The term g represents the source term of the system.

(a) Consider the *transport equation with source term*: $f(u,x,y) = cu$, $c > 0$, and $g(u,x,y) = 2y$. Find a solution. Do the same for a general time dependent source term $g = g(y)$.

(b) Consider the modified Burger's equation in which the flux increases linearly in time: $f(u,x,y) = \frac{u^2}{2}y$, $g \equiv 0$. Set the initial condition $h(x) = 1$ if $x < 0$, $h(x) = 1 - x$ if $x \in [0,1]$ and $h(x) = 0$ if $x > 1$. Find a solution. What is the main difference with the solution of the Burger's equation ($f = \frac{u^2}{2}$)? [1]

SOL:

(a) The ODE system associated to the PDE for a general source term g is

$$\begin{cases} \frac{dx}{dt} = c, & x(0,s) = s, \\ \frac{dy}{dt} = 1, & y(0,s) = 0, \\ \frac{d\tilde{u}}{dt} = g(y), & \tilde{u}(0,s) = h(s). \end{cases}$$

The first two ODEs have solutions $x(t,s) = ct + s$ and $y(t,s) = t$, and from

$$\frac{d\tilde{u}}{dt} = g(y) = g(t),$$

we infer that

$$\tilde{u}(t,s) = \int_0^t g(\tau)\,d\tau + h(s).$$

[1] Try to sketch the characteristic curves

Inverting the variables $t = y$ and $s = x - ct = x - cy$ we get the general solution

$$u(x, y) = h(x - cy) + \int_0^y g(\tau)\, d\tau.$$

Looking at this, we get why g is called the source term: the behaviour is like the one of the classical transport equation, where the transported 'density' described by u as an extra accumulation depending on time described by the integral part of the equation. In particular, when the source is linear $g(y) = 2y$, we get that

$$u(x, y) = h(x - cy) + y^2.$$

(b) The PDE we want to solve is $u_y + yuu_x = 0$ with boundary condition $u(x, 0) = h(x)$. Once again, the ODE system we have to solve is

$$\begin{cases} \frac{dx}{dt} = \tilde{u}y, & x(0, s) = s, \\ \frac{dy}{dt} = 1, & y(0, s) = 0, \\ \frac{d\tilde{u}}{dt} = 0, & \tilde{u}(0, s) = h(s). \end{cases}$$

From $y(t, s) = t$ and $\tilde{u}(t, s) = h(s)$ we get that

$$\frac{dx}{dt} = \tilde{u}y = th(s),$$

giving $x(t, s) = \frac{t^2}{2}h(s) + s$. We need to distinguish three cases:

- When $s < 0$, then $x(t, s) = \frac{y^2}{2} + s$, so that $s = x - \frac{y^2}{2} < 0$,
- when $s \in [0, 1]$ then $x(t, s) = \frac{y^2}{2}(1 - s) + s$, so that $s = (x - y^2/2)/(1 - y^2/2) \in [0, 1]$,
- when $s > 1$, $x = s > 1$.

Putting everything together

$$u(x, y) = \begin{cases} 1, & \text{if } x - y^2/2 < 0, \\ h((x - y^2/2)/(1 - y^2/2)), & \text{if } (x - y^2/2)/(1 - y^2/2) \in [0, 1], \\ 0, & \text{if } x > 1. \end{cases}$$

The difference is purely in the characteristic curves $t \mapsto (x(s, t), y(s, t)) = (t^2/2h(s) + s, t)$, which are parabolas instead of affine lines like in the classical Burger's equation $(x(s, t), y(s, t)) = (th(s) + s, t)$. This is due to the increasing of the flux $f = yu^2/2$ through time that makes the density u flow faster and faster.

5.3. Multiple choice Cross the correct answer(s).

(a) The second order linear PDE given by

$$u_x + x^2 u_{xx} + 2x\sin(y)u_{xy} - \cos^2(y)u_{yy} + e^x = 0,$$

is

X hyperbolic if $x \neq 0$

◯ parabolic in $\{y = k\frac{\pi}{2} : k \in \mathbb{Z}\}$

◯ everywhere hyperbolic

X parabolic in $x = 0$

SOL: Recall that the classification of second order linear PDE is determined by the sign of $\delta := b^2 - ac$. Here $a = x^2$, $b = x\sin(y)$ and $c = -cos^2(y)$. Therefore $\delta = x^2\sin^2(x) + x^2\cos^2(x) = x^2 \geq 0$, with equality at $x = 0$.

(b) Let $A = (a_{ij})$ be a (2×2) real matrix, and $f : \mathbb{R}^2 \to \mathbb{R}$ any smooth function. Then, the PDE [2]

$$\text{Trace}\Big(A \cdot D^2 u\Big) = f,$$

is

X elliptic if A is symmetric and $\det(A) > 0$

◯ hyperbolic if A is antisymmetric and $a_{11} = a_{22} \neq 0$

◯ parabolic if A is symmetric and $\det(A) < 0$

◯ hyperbolic if A is antisymmetric and $a_{11} = -a_{22} > 0$

SOL: We can compute $\text{Tr}(A \cdot D^2 u) = a_{11}u_{xx} + a_{22}u_{yy} + (a_{12} + a_{21})u_{xy}$. Hence, $\delta = (a_{12} + a_{21})^2/4 - a_{11}a_{22}$. Now, if A is symmetric $\delta = a_{12}^2 - a_{11}a_{22} = -\det(A)$, which is strictly less than zero (hence elliptic) iff $\det(A) > 0$. If A is antisymmetric then $a_{11} = a_{22} = 0$ and $\delta = 0$, so the only possibility is A being parabolic, of the form $a_{12}(u_{xy} - u_{yx}) = 0$, but since the second derivatives commutes, we have just the empty expression $0 = f$.

(c) The same options as point **(b)**, but with the PDE

$$\text{div}\Big(A \cdot \nabla u\Big) = f.$$

SOL: The solution is exactly the same as the previous point when one realizes that $\text{div}(A \cdot \nabla u) = (a_{11}u_x + a_{12}u_y)_x + (a_{21}u_x + a_{22}u_y)_y = a_{11}u_{xx} + a_{22}u_{yy} + (a_{12} + a_{21})u_{xy} = \text{Tr}(A \cdot D^2 u)$. Be careful, if A depends on x and y, then this identity is false!

[2] Recall the definition of the Hessian matrix $(D^2u)_{ij} = u_{x_i x_j}$. To start simple: what does it happen when A is the identity matrix?

(d) The following conservation law

$$\begin{cases} u_y + f(u)_x = 0, \\ u(x,0) = c > 0 \text{ for } x < 0 \text{ and } u(x,0) = 0 \text{ for } x \geq 0, \end{cases}$$

has a shock curve of slope equal to 8 if

X $c = 2$ and $f(u) = u^4$

○ $c = 2$ and $f(u) = -u^4$

○ $c = 1$ and $f(u) = u^3$

X $c = 1$ and $f(u) = 2u^2 + 6u - 1$

SOL: Recalling the Rankine-Hugoniot condition, we have that $u^+ = 0$ and $u^- = c$, so that

$$\sigma'(y) = \frac{f^+ - f^-}{u^+ - u^-} = -\frac{1}{c}(f(0) - f(c)) = 8,$$

if and only if $8c = f(c) - f(0)$.

Extra exercises

5.4. Weak solutions II

Consider the equation

$$e^{-u}u_x + u_y = 0,$$

with initial value $u(x,0) = 0$ if $x < 0$, and $u(x,0) = \alpha > 0$ if $x > 0$.

(a) Find a weak solution for any $\alpha > 0$ with a single discontinuity for $y \geq 0$.

(b) Show that such solution fulfils the entropy condition for all $\alpha > 0$.

SOL:

(a) Let us express our equation in the form (2). By inspection, we reach that $F(t) = -e^{-t}$. The curve of discontinuity of our Cauchy problem is then $x = \gamma(y)$ with

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{-e^{-\alpha} + 1}{\alpha} > 0,$$

and our solution is given by

$$u(x,y) = \begin{cases} 0 & \text{if } x < \frac{1-e^{-\alpha}}{\alpha}y \\ \alpha & \text{if } x > \frac{1-e^{-\alpha}}{\alpha}y. \end{cases}$$

(b) The entropy condition is

$$F_u(u_-) > \gamma_y > F_u(u^+), \qquad \text{where} \quad F_u(t) = e^{-t}.$$

That is,

$$1 > \frac{1-e^{-\alpha}}{\alpha} > e^{-\alpha} \quad \iff \quad \alpha > 1 - e^{-\alpha} > \alpha e^{-\alpha}$$

The first inequality holds, since $\frac{d}{d\alpha}\alpha = 1 > e^{-\alpha} = \frac{d}{d\alpha}(1-e^{-\alpha})$ for $\alpha > 0$, and for $\alpha = 0$ they coincide.

The second inequality corresponds to checking

$$f(\alpha) := 1 - e^{-\alpha} > \alpha e^{-\alpha} =: g(\alpha),$$

for $\alpha > 0$. Both sides coincide for $\alpha = 0$, $f(0) = g(0)$. Then, it is enough to check that $f'(\alpha) > g'(\alpha)$ for all $\alpha > 0$. Indeed

$$f'(\alpha) = e^{-\alpha} > e^{-\alpha} - \alpha e^{-\alpha} = g'(\alpha),$$

and we are done.

5.5. Finding shock waves

Consider the transport equation

$$u_y + u^2 u_x = 0,$$

with initial condition $u(x,0) = 1$ for $x \leq 0$, $u(x,0) = 0$ for $x \geq 1$, and

$$u(x,0) = \sqrt{1-x} \quad \text{for} \quad 0 < x < 1.$$

(a) Find the solution using the method of characteristics. Up to which time is the solution defined in a classical sense?

(b) Find a weak solution for all times $y \geq 0$.

SOL:

(a) First notice that the solution has a derivative blowing up for all times $y \geq 0$, since the initial datum is non-smooth. However, we can solve the initial value problem up until the characteristics intersect.

The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = \tilde{u}^2, \quad &&y_t(t,s) = 1, \quad &&\tilde{u}_t(t,s) = 0, \\ &x(0,s) = s, &&y(0,s) = 0, &&\tilde{u}(0,s) = h(s), \end{aligned}$$

where $h(s) = 1$ if $s \leq 0$, $h(s) = 0$ if $s \geq 1$, and $h(s) = \sqrt{1-s}$ otherwise. Notice that $y(t,s) = t$, $\tilde{u}(t,s) = h(s)$, and $x(t,s) = s + h(s)^2 t$.

Let us invert the characteristics:

- If $s \leq 0$, $x(t,s) = s + y(t,s)$, $(t,s) = (y, x-y)$. Note that, since $s \leq 0$, we have $x \leq y$.
- If $0 < s < 1$, $x(t,s) = s + (1-s)t$, and $(t,s) = \left(y, \frac{x-y}{1-y}\right)$. Since $0 < s < 1$, then $0 < \frac{x-y}{1-y} < 1$ implies either $y < x < 1$, or $y > x > 1$.
- If $s \geq 1$, $(t,s) = (y,x)$. In this case, $x \geq 1$.

The characteristics are intersecting at the point $(x,y) = (1,1)$. Thus, the solution is defined up to $y = 1$. For $y \leq 1$, it is:

$$u(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{if} \quad x \geq 1 \\ h\left(\frac{x-y}{1-y}\right) = \sqrt{\frac{1-x}{1-y}} & \text{if} \quad y < x < 1 \end{cases}$$

(b) Let us express the equation in the form

$$u_y + \partial_x \left(F(u)\right) = 0.$$

In this case, a simple inspection yields $F(u) = \frac{1}{3}u^3$.

We are trying to solve $u_y + \partial_x \left(F(u)\right) = 0$ with initial value $u(x,1) = 1$ if $x < 1$, and $u(x,1) = 0$ if $x > 1$.

We just have to compute the slope of the curve of discontinuity, given by

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{1}{3}\frac{0 - 1^3}{0 - 1} = \frac{1}{3}.$$

Thus, the solution for $y \geq 1$ is given by

$$u(x,y) = \begin{cases} 1 & \text{if } x < \frac{1}{3}(y-1) + 1 \\ 0 & \text{if } x > \frac{1}{3}(y-1) + 1, \end{cases}$$

and for $y \leq 1$, as before.