8.1. Separation of variables

Solve the following equations using the method of separation of variables and superposition principle. To do so, write first a general solution solving the problem with boundary conditions, and then impose the initial values.

(a)

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = \sin(2x) + 2\sin(3x) + 4\sin(4x), & x \in [0,\pi]. \end{cases}$$

(b)

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = 2\sin^3(x), & x \in [0,\pi], \\ u_t(x,0) = \sin(4x), & x \in [0,\pi]. \end{cases}$$

Hint: recall that $4\sin^3(x) = 3\sin(x) - \sin(3x)$.

(c)

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u_x(0,t) = 0, & t \in (0,\infty), \\ u_x(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = 1 + \cos(x) & x \in [0,\pi]. \end{cases}$$

SOL:

(a) Assume that $u(x,t) = T(t)X(x)$, for some functions X and T yet to define. Plugging this in the heat equation we get that $T'(t)X(x) = T(t)X''(x)$. Dividing both sides by $T(t)X(x)$ we obtain the identity

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

Since the left hand side depends only x, and the right hand side on t, we infer that there exists $\lambda \in \mathbb{R}$ so that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda.$$

We get the two ODEs

$$T'(x) - \lambda T(x) = 0, \text{ and } X''(x) - \lambda X(x) = 0.$$

The first equation has solution of the form $T(t) = Ae^{\lambda t}$, for some constant $A \in \mathbb{R}$. The second one depends on the sign of λ:

$$X(x) = \begin{cases} B\sin(\sqrt{-\lambda}x) + C\cos(\sqrt{-\lambda}x), & \text{if } \lambda < 0, \\ B\sinh(\sqrt{\lambda}x) + C\cosh(\sqrt{\lambda}x), & \text{if } \lambda > 0, \\ Bx + C, & \text{if } \lambda = 0, \end{cases}$$

for some constants B, C in $\mathbb{R}$. To select the right solution we take advantage of the boundary conditions $u(0,t) = u(\pi,t) = 0$, meaning $X(0) = X(\pi) = 0$. If $\lambda = 0$ we have that $0 = X(0) = C$ and $X(\pi) = \pi B = 0$, implying that $X \equiv 0$. This is not what we are looking for. Same story if $\lambda > 0$: $0 = X(0) = C$ and $0 = X(\pi) = B\sinh(\sqrt{\lambda}\pi)$, imply once again that $X \equiv 0$ since $\sinh(\sqrt{\lambda}\pi) > 0$. Therefore, we are left with the only option $X(x) = B\sin(\sqrt{-\lambda}x) + C\cos(\sqrt{-\lambda}x)$ for some $\lambda < 0$. Now,

$$0 = X(0) = C,$$

implies

$$X(x) = B\sin(\sqrt{-\lambda}x),$$

and

$$0 = B\sin(\pi\sqrt{-\lambda}),$$

implies that if $B \neq 0$, then $\pi\sqrt{-\lambda} = n\pi$ for some $n \in \mathbb{N}$, hence $\lambda = -n^2$. By the superposition principle, we get the formal general solution

$$u(x,t) = \sum_{n\geq 1} D_n e^{-n^2 t}\sin(nx).$$

The only data we have not used yet is the initial condition $u(x,0) = \sin(2x) + 2\sin(3x) + 4\sin(4x)$. Since

$$u(x,0) = \sum_{n\geq 1} D_n \sin(nx),$$

we get that $D_n = 1, 2, 4$ if $n = 2, 3, 4$ respectively, and $D_n = 0$ otherwise, finally getting

$$u(x,t) = e^{-4t}\sin(2x) + 2e^{-9t}\sin(3x) + 4e^{-16t}\sin(4x).$$

(b) Assume that $u(x,t) = T(t)X(x)$, for some functions X and T yet to define. Plugging this in the wave equation we get that $T''(t)X(x) = T(t)X''(x)$. Dividing both sides by $T(t)X(x)$ we obtain the identity

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

Since the left hand side depends only x, and the right hand side on t, we infer that there exists $\lambda \in \mathbb{R}$ so that

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda.$$

We get the two ODEs

$$T''(x) - \lambda T(x) = 0, \text{ and } X''(x) - \lambda X(x) = 0.$$

The solutions depend on the sign of λ:

$$X(x) = \begin{cases} B\sin(\sqrt{-\lambda}x) + C\cos(\sqrt{-\lambda}x), & \text{if } \lambda < 0, \\ B\sinh(\sqrt{\lambda}x) + C\cosh(\sqrt{\lambda}x), & \text{if } \lambda > 0, \\ Bx + C, & \text{if } \lambda = 0, \end{cases}$$

for some constants B, C in $\mathbb{R}$. Since we imposed $u(x,0) = u(\pi,0) = 0$, we select the correct family of solutions exactly as in point (a):

$$X(x) = X_n(x) = B_n \sin(nx).$$

We do the same for T: since $\lambda = -n^2 < 0$ we get

$$T(t) = A_n \sin(nt) + A'_n \cos(nt),$$

obtaining by superposition principle the formal general solution

$$u(x,t) = \sum_{n \geq 1} \sin(nx)\Big(D_n \sin(nt) + D'_n \cos(nt)\Big).$$

By the initial conditions

$$u(x,0) = \frac{3}{2}\sin(x) - \frac{1}{2}\sin(3x),$$

and

$$u_t(x,0) = \sin(4x),$$

since

$$u(x,0) = \sum_{n \geq 1} D'_n \sin(nx),$$

and

$$u_t(x,0) = \sum_{n \geq 1} nD_n \sin(nx),$$

we get that $D'_n = \frac{3}{2}, -\frac{1}{2}$ if $n = 1, 3$ respectively and $D_n = \frac{1}{4}$ if $n = 4$. Finally,

$$u(x,t) = \frac{3}{2}\sin(x)\cos(t) + \frac{1}{4}\sin(4x)\sin(4t) - \frac{1}{2}\sin(3x)\cos(3t).$$

(c) Assume that $u(x,t) = T(t)X(x)$, for some functions X and T yet to define. Plugging this in the heat equation we get that $T'(t)X(x) = T(t)X''(x)$. Dividing both sides by $T(t)X(x)$ we obtain the identity

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

Since the left hand side depends only x, and the right hand side on t, we infer that there exists $\lambda \in \mathbb{R}$ so that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda.$$

We get the two ODEs

$$T'(x) - \lambda T(x) = 0, \text{ and } X''(x) - \lambda X(x) = 0.$$

The first equation has solution of the form $T(t) = Ae^{\lambda t}$, for some constant $A \in \mathbb{R}$. The second one depends on the sign of λ:

$$X(x) = \begin{cases} B\sin(\sqrt{-\lambda}x) + C\cos(\sqrt{-\lambda}x), & \text{if } \lambda < 0, \\ B\sinh(\sqrt{\lambda}x) + C\cosh(\sqrt{\lambda}x), & \text{if } \lambda > 0, \\ Bx + C, & \text{if } \lambda = 0, \end{cases}$$

for some constants B, C in $\mathbb{R}$. To select the right solution we take advantage of the Neumann boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$, meaning $X'(0) = 0$. Now

$$X'(x) = \begin{cases} B\sqrt{-\lambda}\cos(\sqrt{-\lambda}x) - C\sqrt{-\lambda}\sin(\sqrt{-\lambda}x), & \text{if } \lambda < 0, \\ B\sqrt{\lambda}\cosh(\sqrt{\lambda}x) + C\sqrt{\lambda}\sinh(\sqrt{\lambda}x), & \text{if } \lambda > 0, \\ B, & \text{if } \lambda = 0. \end{cases}$$

If $\lambda = 0$ we have the solution $X(x) =$ constant. If $\lambda > 0$ it is easy to check that we have only the trivial solution (similar to point (a)). If $\lambda < 0$ we get that $B = 0$, obtaining the solutions

$$X(x) = C\cos(\sqrt{-\lambda}x).$$

Finally, since $u(x,0) = 1 + \cos(x)$ we get by superposition principle that

$$u(x,t) = 1 + e^{-t}\cos(x).$$

8.2. Multiple choice Cross the correct answer(s).

(a) Let u be solution of the heat equation

$$\begin{cases} u_t - ku_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & x \in (0,L). \end{cases}$$

for $f \in C^\infty(0,T)$. Then, for all $a > 0$

- ○ $\lim_{t\to+\infty} \int_0^L u(x,t)^2\,dx = +\infty$
- X $\lim_{t\to+\infty} t^a \int_0^L u(x,t)^2\,dx = 0$
- X $\lim_{t\to+\infty} \int_0^L u(x,t)^2\,dx = 0$
- ○ $\lim_{t\to+\infty} t^a \int_0^L u(x,t)^2\,dx = +\infty$

SOL: We show this for $L = \pi$, the case with general period $L > 0$ is the same up to rescaling. By the method of separation of variables we have that $u(x,t) = \sum_{n\geq 1} A_n e^{-n^2 t} \sin(nx)$, where A_n are the Fourier coefficients of f, meaning $f(x) = \sum_{n\geq 1} A_n \sin(nx)$. Since

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = \begin{cases} \frac{\pi}{2}, & \text{if } n = m, \\ 0, & \text{otherwise,} \end{cases}$$

we infer that [1]

$$\frac{2}{\pi}\int_0^\pi f(x)^2\,dx = \frac{2}{\pi}\sum_{m,n\geq 1} A_n A_m \int_0^\pi \sin(nx)\sin(mx)\,dx = \sum_{n\geq 1} A_n^2.$$

Similarly,

$$\frac{2}{\pi}\int_0^\pi u(x,t)^2\,dx = \sum_{n\geq 1} e^{-2n^2 t} A_n^2 \leq e^{-2t}\sum_{n\geq 1} A_n^2 = e^{-t}\underbrace{\frac{2}{\pi}\int_0^\pi f(x)^2\,dx}_{\text{constant in } t} \to 0,$$

as $t \to +\infty$. This is still true if we multiply the expression by t^a, since the exponential decreases faster than any polynomial.

(b) Consider the periodic homogeneous wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t) \in [0,1] \times [0,+\infty) \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = 1 + 2021\cos(2\pi x), & x \in [0,1], \\ u_t(x,0) = \cos(40\pi x), & x \in [0,1]. \end{cases}$$

Then, for a fixed point $\bar{x} \in [0,1]$, the function $t \mapsto u(\bar{x},t)$ has period

[1] This is the so called *Parseval's identity*

◯ 1/40 X 1/2

◯ 2π ◯ π

SOL: We have to solve for $u(x,t)$ via separation of variables. Arguing as in Exercise 1, setting $u(x,t) = X(x)T(t)$ we get

$$X''(x) - \frac{\lambda}{4}X(x) = 0, \text{ and } T''(t) - \lambda T(t) = 0,$$

for some constant $\lambda \in \mathbb{R}$. We have possible solutions

$$X(x) = \begin{cases} B\sin(\sqrt{-\lambda}x/2) + C\cos(\sqrt{-\lambda}x/2), & \text{if } \lambda < 0, \\ B\sinh(\sqrt{\lambda}x/2) + C\cosh(\sqrt{\lambda}x/2), & \text{if } \lambda > 0, \\ Bx + C, & \text{if } \lambda = 0, \end{cases}$$

The Neumann boundary conditions $u_x(0,t) = u_x(1,t) = 0$ imply that we have $X(x) =$ contant when $\lambda = 0$, $X \equiv 0$ if $\lambda > 0$ and $X(x) = C\cos(\sqrt{-\lambda}x/2)$ if $\lambda < 0$. From and $X'(1) = 0$, we get that

$$X'(1) = -C\frac{\sqrt{-\lambda}}{2}\sin(\sqrt{-\lambda}/2) = 0,$$

which is possible when $\sqrt{-\lambda}/2 = n\pi$ for $n \in \mathbb{N}$, that is $-\lambda = 4n^2\pi^2$. The ODE for T is then given by

$$T''(t) + 4n^2\pi^2 T(t) = 0,$$

giving $T(t) = T_n(t) = A_n \sin(2n\pi t) + A'_n \cos(2n\pi)$ when $\lambda > 0$, and $T(t) = A_0 t + A'_0$ when $\lambda = 0$. By superposition principle

$$u(x,t) = A_0 t + A'_0 + \sum_{n\geq 1} \cos(n\pi x)\Big(D_n \sin(2\pi nt) + D'_n \cos(2\pi nt)\Big).$$

It is time to use the remaining initial conditions:

$$u(x,0) = 1 + 2021\cos(2\pi x) = A'_0 + \sum_{n\geq 1} D'_n \cos(n\pi x),$$

implies $A'_0 = 1$, $D'_2 = 2021$, and

$$u_t(x,0) = \cos(40\pi x) = A_0 + \sum_{n\geq 1} 2\pi n D_n \cos(n\pi x),$$

implies $A_0 = 0$ and $D_{40} = \frac{1}{80\pi}$. Putting everythig together

$$u(x,t) = 1 + 2021\cos(2\pi x)\cos(4\pi nt) + \frac{1}{80\pi}\cos(40\pi x)\sin(80\pi t).$$