Extra for the first lecture

The "extras" will not contain more material but they are meant to be an additional aid to the comprehension of the material and are mostly based on your questions and feedbacks.

Linear, quasilinear, nonlinear? Let's learn how to recognize these classes of equations:

1. General form of a LINEAR PDE for an unknown function in two independent variables $u(x,y)$:

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u(x,y) = d(x,y)$$

(1st order) a, b, c, d are not dependent on u

2. General form of a second order LINEAR PDE for $u = u(x,y)$

$$a(x,y)\,u_{xx} + b(x,y)\,u_{yy} + 2c(x,y)\,u_{xy} + d(x,y)\,u_x$$
$$+ e(x,y)\,u_y + f(x,y)\,u = g(x,y)$$

3. General form of a QUASILINEAR PDE for $u(x,y)$ (First order)

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y + c(x,y,u) = 0$$

- a and b depend on u but not on u_x and/or u_y.

c does not depend on u_x and or u_y.

4. General form of a second order quasi-linear PDE (it is linear in the highest order derivatives, (in this case second order))

$$a(x,y,u,u_x,u_y)u_{xx} + b(x,y,u_x,u_y)u_{yy} +$$
$$+2c(x,y,u,u_x,u_y)u_{xy} = d(x,y,u,u_x,u_y)$$

A nonlinear equation <u>can</u> be quasilinear if it is linear in the highest order terms.
A quasilinear equation <u>IS</u> a nonlinear equation <u>but</u> with a good property of being linear w.r.t. the highest terms, that can be thought as "dominant".

Examples:

- $u_x + u_{tt} + 3u = e^x$ — non homogeneous, 2nd order linear
- $u_x + u_{tt} + 3uu_x = e^x$ — 2nd order, nonlinear but quasilinear (u_{tt} appears linearly)
- $u_{xx} + (u_t)^2 + e^u = 0$ — 2nd order, non linear but quasilinear because u_{xx} appears linearly
- $(u_{xx})^2 + u_t + e^u = 0$ — 2nd order, nonlinear because u_{xx} appears non-linearly

- $\sin(u) + u_x + \overset{\text{appears linearly}}{u_{yy}} = 0$ — second order, quasilinear

- $u + u_x + \underbrace{\sin(u_{yy})}_{\text{nonlinear term}} = 0$ — second order, nonlinear

- $u\sin(u_x) + \underbrace{u_{yy}}_{\text{linear term}} = 0$ — second order, nonlinear but quasilinear

- $\underbrace{(u_x)^2 + (u_y)^2}_{\text{nonlinear terms BUT first order}} + \underbrace{u_{xy}}_{\text{linear second order}} = 0$ — second order, quasilinear.

- $\det(D^2 u) = f$ — second order, nonlinear

  Monge-Ampère eq.

  the determinant is MULTILINEA[R] and linear only for matrices of side 1.

- $|\nabla u| = f$ — 1st order, nonlinear

  $$= \sqrt{(u_x)^2 + (u_y)^2}$$

- $u_t + \operatorname{div}(\underline{v}\, u) = \Delta u + f$ — second order, non hom, linear

  $\underline{v}$ vector $(v_1, .., v_n)$

  divergence, laplacian and gradient are linear operators!

  $$\sum_i v_i u_{x_i}$$