Some comments on harmonic and holomorphic functions (not examinable)

I recall that, given a complex-valued function f of a single complex variable $z = x+iy$, the derivative of f at a point z_0 in its domain is defined by the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If f is differentiable at every point z_0 in a domain D we say that f is holomorphic on D.

If a complex function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic, then u and v have first partial derivatives w.r.t x and y, and satisfy the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

It is a natural question whether a converse of this statement holds. The following is true: if u and v have continuous first partial derivatives and satisfy the Cauchy-Riemann equations, then f is holomorphic.

The key connection between harmonic functions (i.e. solutions of the Laplace equation) and holomorphic functions is that both the real and the imaginary parts of holomorphic functions are harmonic. This is a consequence of the Cauchy-Riemann equations: Since $u_x = v_y$ we have $u_{xx} = v_{yx}$. Likewise, $u_y = -v_x$ implies $u_{yy} = -v_{xy}$. Since $v_{xy} = v_{yx}$ we have

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

Moreover it is possible to prove that any harmonic function is the real part of a holomorphic function.

Maximum principle and mean value property:

- Mean value property: If u is harmonic then it satisfy the mean value property. That is, suppose u is harmonic on $\overline{B_R(z_0)}$ with $z_0 = x_0 + iy_0$. Then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + R e^{i\theta})\, d\theta .$$

Proof. Let $f = u + iv$ be a holomorphic function with u as real part. The mean value property for holom. functions says that

$$\underbrace{u(x_0, y_0) + i v(x_0, y_0)}_{f(x_0 + iy_0)} = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + R e^{i\theta})\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[u(z_0 + R e^{i\theta}) + i v(z_0 + R e^{i\theta}) \right] d\theta .$$

The statement follows by looking at the real part of this equality. □

The proof of the maximum principle for maximum points is identical to the one for the maximum modulus principle (if f is holomorphic on some connected open subset D of the complex plane $\mathbb{C}$ with complex values, if $z_0 \in D$ s.t $|f(z_0)| \geq |f(z)|$ for all z in a neighborhood of z_0, then f is constant in D).

For more info, look at "Real and complex analysis" by W. Rudin, chapter 11.