Extra 10. 10. 2019

This extra lecture notes aim to clarify some points already discussed in class following some questions I received.

Question 1: Why the characteristic equations are exactly given by the formula (*) ?

Answer 1: Let me focus for now on the simplest case: 1st order PDEs in two variables, linear.

Consider the equation:

$$(1) \quad a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)$$

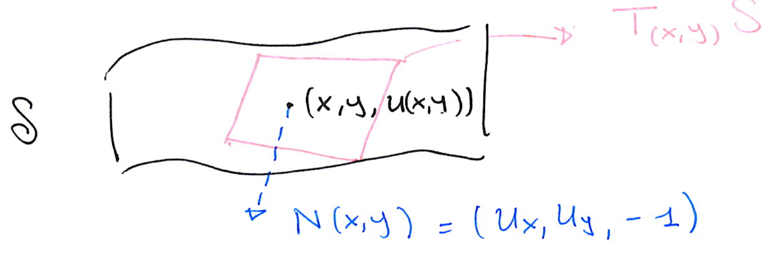
Suppose that $u(x,y)$ is a solution to (1). Then we can consider the surface in $\mathbb{R}^3$ given by its graph:

$$S \equiv \text{graph}(u) = \{(x,y,u(x,y))\} \subseteq \mathbb{R}^3 .$$

If u is a solution of (1) at each point (x,y) in its domain the solution verifies:

$$(a(x,y),\, b(x,y),\, c(x,y)) \cdot (u_x(x,y),\, u_y(x,y),\, -1) = 0$$

Therefore the vector $(a(x,y), b(x,y), c(x,y))$ lies in the tangent plane to S at the point (x,y).



Therefore to find a solution to (1) we look for a surface S such as at each point $(x,y,u) \subset S$ the vector $(a(x,y), b(x,y), c(x,y)) \in T_{(x,y,u)} S$.

Let's ℓ be a curve parametrized by t such that at each point on ℓ the vector:

$$(a(x(t),y(t)), b(x(t),y(t)), c(x(t),y(t)))$$

is tangent to the curve.

In particular, the curve $\ell = \{(x(t),y(t),\tilde{u}(t))\}$ will satisfy the following system of ODEs:

$$(*) \begin{cases} \dfrac{dx}{dt} = a(x(t),y(t)) \\[2mm] \dfrac{dy}{dt} = b(x(t),y(t)) \\[2mm] \dfrac{d\tilde{u}}{dt} = c(x(t),y(t)) \quad , \quad \boxed{u(x(t),y(t)) = \tilde{u}(t)} \end{cases}$$

This set of equations is known as the set of characteristic equations for (1).

Once we found the characteristic curves for (1) we wish to form the solution surface S as a union of those curves.

Indeed we reduced our PDE to a system of ODEs.

Of course, when solving a Cauchy problem (PDE + Initial condition) we need to take into account the initial condition as we have seen in class (see Lecture 2 and Lecture 3).

Alternatively, if one does not want to use the geometric interpretation, one can think as follows:

Let u be a solution of (1):

$$a(x,y)u_x + b(x,y)u_y = c(x,y).$$

Let $t \mapsto (x(t), y(t))$ be a curve that solves:

$$(**) \begin{cases} \dfrac{dx}{dt} = a(x(t), y(t)) \\[2ex] \dfrac{dy}{dt} = b(x(t), y(t)) \end{cases}$$

and consider $t \mapsto u(x(t), y(t))$. Then

$$\frac{d}{dt}\left[u(x(t), y(t))\right] \underset{\uparrow}{=} \dot{x}u_x + \dot{y}u_y \underset{\nwarrow}{=} au_x + bu_y \overset{(1)}{=} c(x(t), y(t))$$

$$\left(\text{chain rule}, \ \dot{x} = \frac{dx}{dt}\right) \qquad (**)$$

In other words, u along the curve $(x(t), y(t))$ coincides with the solution of the ODE

$$\frac{d\tilde{u}(t)}{dt} = c(x(t), y(t))$$

(Provided, of course, they start from the same initial condition)

Question 2: What is the local existence thm for ODEs saying?

Answer 2: Consider the initial value problem

$$(2)\quad \begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

We wish to understand:

a) when (2) has a solution,

b) if this solution is unique.

Then, using the Picard-Lindelof (also named Cauchy-Lipschitz) theorem we have an answer to question a) and b).

Thm (Picard-Lindelof / Cauchy-Lipschitz) Suppose that f in (2) is uniformly Lipschitz continuous in y (meaning that the Lipschitz constant can be taken independently of t) and continuous in t, then for some $\varepsilon > 0$ there exists a unique solution $y(t)$ to (2) on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

Recall: f is uniformly Lipschitz continuous in y if

$$|f(t, y_1) - f(t, y_2)| \leq M |y_1 - y_2|$$

Note that the existence part of the theorem (a) holds under much weaker assumptions: see for instance Peano's thm that guarantees local existence of a solution if f is continuous.