Bonus Guidelines. You can choose three exercises from the ten exercises below. Each correct solution earns you 1 point, while an incorrect one yields 0 points. Please note:

1. You're welcome to write your solutions in either German or English.
2. Minor computational errors are accepted, as long as they don't simplify the exercise.
3. **Submitting more than three exercises will raesult in disregarding ALL exercises (earning zero points).**
4. Use the "Bonus" option on the SAM-Up tool to submit your solutions. Ensure that **you are connected to an ETH WiFi or using a VPN**.
5. Make sure to upload your solutions before Wednesday 13.12. at 12:00.

The bonus added to your grade (before rounding) follows the formula:

Points	Bonus
1	0.1
2	0.2
3	0.25

Table 1: Conversion Table for Points to Bonus.

Exercise 1. Compute

$$a := \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}.$$

i.e. the limit of the recursive sequence $(a_n)_{n\in\mathbb{N}}$ with $a_0 = 1$ and $a_{n+1} = \sqrt{1+a_n}$.

Clarification: You also need to show that the sequence converges.

Exercise 2. Show the equivalence of the following two statements:

i) Every Cauchy sequence in $\mathbb{R}$ converges.

ii) Every absolutely convergent series in $\mathbb{R}$ is convergent.

Clarification: It is not sufficient to say that i) was proved in the lecture. You need to show that i) implies ii) and that ii) implies i).

Exercise 3. The exercises a) and b) are independent.

a) In this exercise, we want to show the following statement :

   Let $a, b \in \mathbb{R}$ with $a < b$ and assume that $f : [a, b] \to \mathbb{R}$ is differentiable. Show that for every real number $y \in \mathbb{R}$ between $f'(a)$ and $f'(b)$, there is a $c \in [a, b]$ such that $f'(c) = y$.

   Note that we only assume that f' exists, not that it is continuous. To prove this result, proceed as follows:

   i) Assume that $f'(a) > 0$ and $f'(b) < 0$. Show that there exists a $c \in [a, b]$ such that $f'(c) = 0$.

      Hint: Look for a Maximum.

   ii) Assume that $f'(a) > f'(b)$ and show that for each real number $y \in (f'(b), f'(a))$, there is a $c \in [a, b]$ such that $f'(c) = y$.

   iii) Show the statement.

b) i) Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable such that $f' > 0$. Show that f is injective.

ii) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative is bounded on $\mathbb{R}$ by a constant $M > 0$ i.e. $|g'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that, for $a \in (-\frac{1}{M}, \frac{1}{M})$, the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x + a \cdot g(x) \end{aligned}$$

is injective.

Exercise 4.

a) Let I be a compact (i.e. closed and bounded) non-empty interval and $f : I \to \mathbb{R}$ a continuous function such that $f(I) \subseteq I$. Show that there is an $x \in I$ such that $f(x) = x$.

b) Show that statement a) is false if we assume that I is closed, but not necessarily bounded.

c) Show that statement a) is false if we assume that I is bounded, but not necessarily closed.

Exercise 5.

a) Determine where the following functions $f : \mathbb{R} \to \mathbb{R}$ are discontinuous:

i) $f(x) = \frac{4x+5}{9-3x}$

ii) $f(x) = \frac{6}{x^2-3x-10}$

iii) $f(x) = \frac{9x^2+102x+289}{3x+17}$

iv)
$$f(x) = \begin{cases} 1 - 3x & x < -6 \\ 7 & x = -6 \\ x^3 & -6 < x < 1 \\ 1 & x = 1 \\ 2 - x & x > 1. \end{cases}$$

v) $f(x) = \frac{1}{2-4\cos(\frac{x}{3})}$

Clarification: You also have to investigate what happens at the points where f might not be defined. Is there a continuous extension of the function?

b) Show that there exists at least one solution to the following equations in the indicated interval:

   i) $w^2 - 4\log(5w+2) = 0$ on $[0,4]$,

   ii) $4t + 10e^t - 2e^{2t} = 0$ on $[1,3]$.

c) Let $f_n : [0,1] \to \mathbb{R}$ be the sequence of functions given by

$$f_n(x) = \frac{n^2 x}{nx^2 + n^2 x + 1}.$$

Does this sequence of functions converge pointwise or uniformly? If possible, determine the limit.

Exercise 6.

a) Let $z \in \mathbb{C}$ be a complex number. Analyse the convergence behaviour of the series $\sum_{n=0}^{\infty} z^n$ and calculate the limit if it exists.

b) Let $\theta \in \mathbb{R}$. Analyse the convergence behaviour of the series $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$ and calculate the limit if it exists.

Hint: You may use without proof that for $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} \mathrm{Re}\,(a_n) = \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n\right).$$

Exercise 7.

a) Show that for any $\gamma > 0$ the "reciprocal" function $g : (\gamma, \infty) \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is uniformly continuous.

b) Show that the function $h : \mathbb{R}_{>0} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is not uniformly continuous on the non-negative reals.

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function.

a) Prove that the right and left derivative exist at every point.

b) Show that f is continuous.

Hint: Use part a).

Exercise 9. Compute the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

proceeding as follows:

a) For every $n \in \mathbb{Z}$, compute the integral

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos(nx) - i \sin(nx))\, dx$$

for $f(x) = x$.

b) Compute the value of the series using Parseval's identity (which you do not have to prove!)

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_{-n}|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx.$$

Exercise 10. Consider the improper integral

$$\int_0^{\infty} \frac{\sin(x)}{x^{\alpha}} dx$$

with $\alpha > 0$. For which values of α does it converge?